The Energetics of Baroclinic Waves

We now consider the energetics of linearized baroclinic disturbances and shows that the available potential energy of the mean flow is the energy source for the growth of baroclinic unstable perturbations.

Available potential energy

In a hydrostatic atmosphere, the gravitational potential energy and internal energy are proportional with each other and these two forms of energy can be combined to form a single term called total potential energy.

Internal Energy:

\[ dE_I = \rho c_v T \, dz \]
\[ E_I = \int_0^z \rho T \, dz \]

Gravitational potential energy:

\[ dE_p = \rho gz \, dz \]
\[ E_p = \int_0^z \rho gz \, dz = -\int_0^z dzp = -\int_0^z d(zp) + \int_0^z pdz \]
\[ = \int_0^z pdz = R \int_0^z \rho T \, dz \]

Comparing (8.29) and (8.31) we see that \( \rho c_v E_p = c_p E_I \). Thus, the total potential energy may be expressed as
\[ E_p + E_I = \left( \frac{c_p}{c_v} \right) E_I = \left( c_p / R \right) E_p \]

Therefore, in a hydrostatic atmosphere the total potential energy can be obtained by computing either \( E_I \) or \( E_p \) alone.

But note that only a small portion of the total potential energy can harnessed for the conversion to kinetic energy. We demonstrate this point in a very simple model.

We consider the difference results from adiabatic rearranging two equal masses (different temperature) of dry air initially separated by a vertical partition toward a minimum possible total potential energy (Fig. 8.7).
Initially the TPE is:

\[ \left( \frac{c_p}{c_v} \right) E_i \]

The minimum TPE is realized when the heavier air mass lies entirely beneath the light air:

\[ \left( \frac{c_p}{c_v} \right) E_i'' \]

In this case the total potential energy \( \left( \frac{c_p}{c_v} \right) E_i'' \) is not available for conversion to kinetic energy because no adiabatic process can further reduce \( \left( \frac{c_p}{c_v} \right) E_i'' \).

The available potential energy (APE) can now be defined as the difference between the total potential energy of a closed system and the minimum total potential energy that could result from an adiabatic redistribution of mass. For this idealized model, the APE, designated by symbol \( P \), is

\[ P \equiv \left( \frac{c_p}{c_v} \right) (E_i - E_i'') \] (8.36)

which is equivalent to the maximum kinetic energy that can be realized by an adiabatic process.

It can be proved (Lorenz, 1960) that the APE is given approximately by the volume integral over the entire atmosphere of the variance of potential temperature on isobaric surfaces, i.e.,

\[ P \propto V^{-1} \int \left( \theta''^2 / \bar{\theta}^2 \right) dV \]

where \( \bar{\theta} \) is the average potential temperature for a given pressure surface and \( \theta'' \) the local deviation from the average. \( V \) designates the total volume. For the QG model, this proportionality is an exact measure of the APE.
Observations indicate that for the atmosphere as a whole,
\[ \frac{P}{(c_p/c_v)E_I} = 5 \times 10^{-3}, \quad \frac{K}{P} \approx 10^{-4} \]
Thus only about 0.5% of the total potential energy of the atmosphere is available, and of the available portion only about 10% is actually converted to kinetic energy. From this point of view the atmosphere is a rather inefficient heat engine.

Energy equations for the two-layer model

The energy equations for the two-layer model introduced for the discussion of baroclinic instability above can be derived by manipulating equations (8.9)-(8.11) and integrating over one wavelength of the perturbation in the zonal direction.

With the definitions of the two-layer model’s perturbation (eddy) kinetic energy
\[ K' \equiv (1/2) \left[ (\partial \psi_{1}'/\partial x)^2 + (\partial \psi_{3}'/\partial x)^2 \right] \]
and perturbation (eddy) available potential energy
\[ P' \equiv \lambda^2 \left( |\psi_{1}' - \psi_{3}'| \right)^2 / 2, \]
(remember the definition of \( \lambda^2 \) for the two-layer model)
we have a pair of energy equations for the perturbation kinetic energy and potential energy:
\[ \frac{dK'}{dt} = \left[ -(f_0 / \delta p) \omega_2' \left( \psi_{1}' - \psi_{3}' \right) = -(2f_0 / \delta p) \omega_2' \psi_T \right] \]
\[ \frac{dP'}{dt} = \lambda^2 U_f \left( \psi_{1}' - \psi_{3}' \right) \frac{\partial \psi_{1}' + \psi_{3}'}{\partial x} + (f_0 / \delta p) \omega_2' \left( \psi_{1}' - \psi_{3}' \right) \]
\[ = 4\lambda^2 U_f \psi_T \frac{\partial \psi_m}{\partial x} + (2f_0 / \delta p) \omega_2' \psi_T \]
\[ (8.37) \]
\[ (8.38) \]
The last term in (8.38) is just equal and opposite to the kinetic energy source term in (8.37). This term clearly must represent a conversion between potential and kinetic energy. If on average, the vertical motion is positive (\( \omega_2' < 0 \)) where the thickness is greater than average (\( \left| \psi_{1}' - \psi_{3}' \right| > 0 \)) or vertical motion if negative where thickness is less than average, then
\[ \omega_2' \left( \psi_{1}' - \psi_{3}' \right) = 2\omega_2' \psi_T < 0 \]
and perturbation potential energy is being converted to kinetic energy. Physically, this correlation represents an overturning in which cold air aloft is replaced by warm air from below, a situation that clearly tends to lower the center of mass and hence the potential energy of the perturbation. However, the eddy available potential energy and kinetic energy can still grow simultaneously, provided that the potential energy generation due to the first term in (8.38) exceeds the rate of eddy potential energy conversion to eddy kinetic energy.

The first term on the rhs of (8.38) is the generation term of eddy potential energy, which depends on the correlation between the perturbation thickness \( \psi_T \) and the meridional velocity at 500 hPa, \( \frac{\partial \psi_m}{\partial x} \). To understand the role of this term, it is helpful to
consider a particular sinusoidal wave disturbance. Suppose that the barotropic and baroclinic parts of the disturbance can be written, respectively, as

\[ \psi_m = A_m \cos(k(x - ct)) \quad \text{and} \quad \psi_T = A_T \cos(k(x + x_0 - ct)) \quad (8.39) \]

where \( x_0 \) designates the phase difference. Thus, \( kx_0 \) indicates the phase difference of 500 hPa temperature lagging 500 hPa geopotential. Using (8.39), the generation term

\[
\psi_T \frac{\partial \psi_m}{\partial x} = \frac{k}{L} \int_0^L A_T A_m \cos(k(x + x_0 - ct)) \sin(k(x - ct)) dx
\]

\[
= \frac{kA_T A_m \sin(kx_0)}{L} \int_0^L [\sin(k(x - ct))]^2 dx
\]

\[(8.40)\]

Substituting from (8.40) into (8.38) we see that for the usual midlatitude case of a westerly thermal wind \( (U_T > 0) \) the correlation in (8.40) must be positive if the perturbation potential energy is to increase. Thus \( kx_0 \) must satisfy the inequality

\[ 0 < kx_0 < \pi. \]

Furthermore, the correlation will be positive maximum for \( kx_0 = \pi / 2 \), that is, when the temperature field lags the geopotential field by 90 deg in phase at 500 hPa. This case is shown schematically in Fig. 8.4.

**Fig. 8.4** Structure of an unstable baroclinic wave in the two-level model. (Top) Relative phases of the 500-hPa perturbation geopotential (solid line) and temperature (dashed line). (Bottom) Vertical cross section showing phases of geopotential, meridional temperature advection, ageostrophic circulation (open arrows), Q vectors (solid arrows), and temperature fields for an unstable baroclinic wave in the two-level model.
For the unstable perturbations, temperature wave lags the geopotential by one-quarter cycle.

The northward advection of warm air by the geostrophic wind east of the 500-hPa trough and the southward cold advection west of 500-hPa trough are both maximized.

The upper-level disturbance will intensify.

If the temperature wave lags the geopotential wave, the trough and ridge axes will tilt westward with height.

Vertical motion must be downward in the cold air behind the trough at 500 hPa. Hence, \( \frac{\partial}{\partial t} \psi < 0 \)

Therefore, the westward tilt implies that the horizontal temperature advection will increase the available potential energy of the perturbation and that the vertical circulation will convert perturbation potential energy to perturbation kinetic energy.

Potential energy generation rate determines the growth of the total energy \( P' + K' \). This may be proved by adding (8.37) and (8.38):

\[
\frac{d(P' + K')}{dt} = 4 \pi^2 U_1 \psi_{\psi m} \frac{\partial \psi_m}{\partial x}
\]

and the \( \frac{\partial}{\partial t} \psi_i \) term merely converts disturbance energy between the available potential and kinetic forms without affecting the total energy of perturbation.

The rate of increase of total energy of perturbation depends linearly on the magnitude of the background mean thermal wind.

Because the generation of perturbation energy requires systematic poleward transport of warm air and equatorward transport of cold air, baroclinically unstable disturbances tend to reduce the meridional temperature gradient and hence the available potential energy of the mean flow.
In order that perturbations are able to extract potential energy from the mean flow, the perturbation parcel trajectories in the meridional plane must have slopes less than the slopes of the mean potential temperature surfaces, and a permanent rearrangement of air must take place for there to be a net heat transfer.

Because the slopes of the parcel trajectories increase as the wavelength decreases, for some critical wavelength the trajectory slopes will become greater than the slopes of the potential temperature surfaces. Therefore waves shorter than the critical wavelength cannot grow baroclinically and this is why baroclinic instability has a short-wave cutoff.

Fig. 8.8 Slopes of parcel trajectories relative to the zonal mean potential temperature surfaces for a baroclinically unstable disturbance (solid arrows) and for a baroclinically stable disturbance (dashed arrows).
The energy flow for amplifying QG perturbations is summarized in Fig. 8.9

\[
\left\langle \left( \psi_1^2 - \psi_3^2 \right) \frac{\partial}{\partial x} \left( \psi_1^2 + \psi_3^2 \right) \right\rangle
\]

\[
P' \quad \text{Perturbation available potential energy}
\]

\[
\left\langle \omega'_x \left( \psi_1^2 - \psi_3^2 \right) \right\rangle \quad \rightarrow \quad K'
\quad \text{Perturbation kinetic energy}
\]

**Fig. 8.9** Energy flow in an amplifying baroclinic wave.

**The Lorenz Energy Cycle --- for QG flow on \( \beta \)-plane**

1. QG hydrostatic flow on midlatitude on \( \beta \)-planes
2. log-\( p \) coordinates
3. neglect vertical eddy fluxes and advection by \( \bar{v} \)

\[
\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v} = - \frac{\partial \left( \bar{u} \bar{v}' \right)}{\partial y} + \bar{X}
\]

\[
f_0 \bar{u} = - \frac{\partial \Phi'}{\partial y}
\]

\[
\frac{\partial}{\partial z} \left( \frac{\partial \Phi'}{\partial z} \right) + \bar{N}^2 = \frac{\kappa}{H} \bar{J} - \frac{\partial}{\partial y} \left( \bar{v} \frac{\partial \Phi'}{\partial z} \right)
\]

(10.47-50)

\[
\frac{\partial \bar{v}}{\partial y} + \rho'_0 \frac{\partial \bar{u}}{\partial z} = 0
\]

Note here the overbar indicates zonal average (not time average).

To analyze the exchange of energy between mean flow and eddies, we require a similar set of dynamical equations for the eddy motion. The following equations are derived by subtracting the zonal mean equations (10.47-50) from the original QG equation system.
\[
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) u' - \left( f_0 - \frac{\partial \bar{u}}{\partial y} \right) v' = -\frac{\partial \Phi'}{\partial x} + X' \\
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) v' + f_0 u' = -\partial \Phi'/\partial y + Y' \\
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \frac{\partial \Phi'}{\partial z} + v' \frac{\partial}{\partial y} \left( \frac{\partial \Phi}{\partial z} \right) + N^2 w' = \frac{\kappa}{H} J'
\]

\[
\partial u'/\partial x + \partial v'/\partial y + \rho_0 \frac{\partial (\rho_0 w')}{\partial z} = 0
\]

where \( X' \) and \( Y' \) are the zonally asymmetric components of drag due to unresolved turbulent motions.

We now define a global average
\[
\langle \cdot \rangle = \frac{1}{A} \int \int \int \langle 0 \rangle \, dx \, dy \, dz
\]

where \( L \) is the distance around a latitude circle, \( D \) is the meridional extent of the midlatitude beta plane, and \( A \) designates the total horizontal area of the beta plane. Then for any quantity \( \Psi \)
\[
\langle \frac{\partial \Psi}{\partial x} \rangle = 0 \\
\langle \frac{\partial \Psi}{\partial y} \rangle = 0, \text{ if } \Psi \text{ vanishes at } y = \pm D \\
\langle \frac{\partial \Psi}{\partial z} \rangle = 0, \text{ if } \Psi \text{ vanishes at } z = 0 \text{ and } z \rightarrow \infty
\]

An equation for the evolution of the mean flow kinetic energy can then be obtained by multiplying (10.47) by \( \rho_0 \bar{u} \) and (10.48) by \( \rho_0 \bar{v} \) and adding the results to get
\[
\rho_0 \frac{\partial}{\partial t} \left( \begin{array}{c}
\bar{\mu}^2 \\
\end{array} \right) = -\rho_0 \bar{v} \frac{\partial \Phi}{\partial y} - \rho_0 \bar{u} \frac{\partial}{\partial y} \left( \bar{u} \bar{v}' \right) + \rho_0 \bar{u} \bar{\Phi}' \\
= -\frac{\partial}{\partial y} \left( \rho_0 \bar{\Phi}' \right) + \rho_0 \bar{\Phi}' \frac{\partial \bar{\mu}}{\partial y} + \rho_0 \bar{u} \bar{v}' \frac{\partial \bar{\mu}}{\partial y} + \rho_0 \bar{u} \bar{\Phi}'
\]

After integration over the entire volume we get the mean kinetic energy equation
\[
\frac{d}{dt} \left( \rho_0 \bar{\mu}^2 \right) = + \left( \rho_0 \bar{\Phi}' \frac{\partial \bar{\mu}}{\partial y} \right) + \left( \rho_0 \bar{u} \bar{v}' \frac{\partial \bar{\mu}}{\partial y} \right) + \left( \rho_0 \bar{u} \bar{\Phi}' \right)
= \frac{R}{H} \left( \rho_0 \bar{\Phi}' \right) + \left( \rho_0 \bar{u} \bar{v}' \frac{\partial \bar{\mu}}{\partial y} \right) + \left( \rho_0 \bar{u} \bar{\Phi}' \right)
\]

(10.55)

Note the equivalency between the work done by zonal-mean pressure force and the term of vertical flux of mean temperature by mean vertical motion. The 2nd and 3rd terms represent, respectively, the conversion of eddy kinetic energy to zonal-mean kinetic energy, and dissipation by the zonal-mean eddy stress.

If we define an area-mean, zonal-mean available potential energy
\[ \bar{P} = \frac{1}{2} \left( \rho_0 \left( \frac{\partial \Phi}{\partial z} \right)^2 \right) \]

Multiplying (10.49) through by \( \rho_0 \left( \partial \Phi / \partial z \right) / N^2 \) and averaging over space gives

\[ \frac{d}{dt} \left( \frac{\rho_0}{2N^2} \left( \frac{\partial \Phi}{\partial z} \right)^2 \right) = -\left( \rho_0 \frac{\partial}{\partial z} \left( \frac{\partial \Phi}{\partial z} \right) \right) \]

The first term on the rhs is just equal and opposite to the first term on the rhs of equation (10.55), which confirms that this term represents a conversion between zonal-mean kinetic and potential energies. The second term involves the correlation between temperature and diabatic heating; it expresses the generation of zonal-mean potential energy by diabatic processes. The final term, which involves the meridional eddy heat flux, expresses the conversion between zonal-mean and eddy potential energy.

With the eddy kinetic energy and potential energy defined, respectively, as

\[ K' = \rho_0 \left( \frac{u'^2 + v'^2}{2} \right), \quad P' = \frac{1}{2} \left( \rho_0 \left( \frac{\partial \Phi'}{\partial z} \right)^2 \right) \]

one can readily obtain equations for the eddy kinetic energy and available potential energy.

\[ \frac{dK'}{dt} = + \left\{ \rho_0 X + \frac{\partial}{\partial y} \left( \frac{\partial u'}{\partial y} \right) \right\} + \left\{ \rho_0 \left( \frac{u'X + v'Y}{2} \right) \right\} \]

\[ \frac{dP'}{dt} = - \left\{ \rho_0 \frac{\partial}{\partial z} \left( \frac{\partial \Phi'}{\partial z} \right) \right\} + \left\{ \rho_0 \left( \frac{u'X + v'Y}{2} \right) \right\} \]

Expressing the conversion terms and the sink and source terms compactly

\[ \left[ \bar{P} \cdot \vec{K} \right] = \rho_0 w \frac{\partial \Phi}{\partial z}, \quad \left[ P' \cdot K' \right] = \left( \rho_0 \left( \frac{\partial \Phi'}{\partial z} \right) \right), \]

\[ \left[ K' \cdot \vec{K} \right] = \left\{ \rho_0 \left( \frac{u'v' - \partial u}{\partial y} \right) \right\}, \quad \left[ P' \cdot P' \right] = \left\{ \rho_0 \left( \frac{v' \left( \frac{\partial \Phi'}{\partial z} \right)}{N^2} \right) \right\}, \]

\[ \bar{R} = \rho_0 \frac{\partial}{\partial z} \left( \frac{\partial \Phi}{\partial z} \right), \quad \vec{R}' = \frac{\rho_0 \partial^2 \Phi}{N^2 \partial z}, \]

\[ \bar{e} = \rho_0 \left( \frac{u'X + v'Y}{2} \right), \quad e' = \left( \rho_0 \left( \frac{u'X + v'Y}{2} \right) \right) \],

then we can write equations (10.55)-(10.58) in the simple form
\[ \begin{align*}
\frac{d\bar{K}}{dt} &= \left[ \bar{P} \cdot \bar{K} \right] + \left[ \bar{K} \cdot \bar{K} \right] + \bar{e} \\
\frac{d\bar{P}}{dt} &= -\left[ \bar{P} \cdot \bar{K} \right] + \left[ \bar{P}' \cdot \bar{P} \right] + \bar{R} \\
\frac{dK'}{dt} &= \left[ P' \cdot K' \right] - \left[ \bar{K} \cdot \bar{K} \right] + \epsilon' \\
\frac{dP'}{dt} &= -\left[ P' \cdot K' \right] - \left[ P' \cdot \bar{P} \right] + R'
\end{align*} \]  

Adding (10.59)-(10.62), we find that the rate of change of total energy
\[ \frac{d}{dt} (\bar{K} + K' + \bar{P} + P') = \bar{R} + R' + \bar{e} + \epsilon' \]  

Thus, for adiabatic inviscid flow the total energy is conserved.

In the long term mean the lhs of (10.63) must vanish. Thus, the production terms must balance the dissipation terms:
\[ \bar{R} + R' = -\bar{e} - \epsilon' \]  

It is obvious that \( \bar{R} > 0, \bar{e} < 0, \epsilon' < 0 \). For a dry atmosphere in which eddy diabatic processes are limited to radiation and diffusion (at sub-grid scales), the diabatic production of eddy available potential energy should be negative because the thermal radiation emitted to space from the atmosphere increases with increasing temperature and thus tends to reduce the horizontal temperature contrasts in the atmosphere. However, for the Earth’s atmosphere, the presence of clouds and precipitation greatly alters the distribution of \( R' \). Our estimates (see Fig.10.13) suggest that in the Northern Hemisphere \( R' \) is positive and nearly half as large as \( \bar{R} \).
The zonal-mean diabatic heating generates mean zonal available potential energy through a net heating of the tropics and cooling of the polar regions.

Baroclinic eddies transport warm air poleward, cold air equatorward, and transform the mean available potential energy to eddy available potential energy (this is paradoxical depending the diagnosis framework).

At the same time eddy available potential energy is transformed into eddy kinetic energy by the vertical motions in the eddies.

The zonal kinetic energy is maintained primarily by the conversions from eddy kinetic energy due to the correlation $u'v'$.

The energy is dissipated by surface and internal friction in the eddies and mean flow.

Note that only about 10% of the mean available potential portion is actually converted to mean kinetic energy----the atmosphere is an inefficient heat engine. But for the eddies, the ratio is 50%.

A note of caution should be taken that the conversion terms and direction of conversion given here are result of the particular type of zonal average model used. The analogous energy equations for the TEM equations have rather different conversions and the attendant different interpretation (Plumb, 1983; Kanzawa, 1984).
Fig. 3. Comparisons of energy cycles of the global atmosphere among (top) the $p$, analysis, (bottom left) the conventional Eulerian mean, and (bottom right) the TEM.

\[ PZ \equiv \bar{P}, \quad KZ \equiv \bar{K} \]
\[ PE \equiv P', \quad KE \equiv K' \]

Some of studies have questioned the classical interpretation of the energy cycle (Lorenz, 1955). In particular, it has been pointed out that the individual conversion terms and flux vectors are not good indicators of wave propagation or generation characteristics (McIntyre 1980; Plumb, 1983; Hulst, 2001). Plumb (1983) reviewed the traditional derivation of the energy cycle and noted that the paradoxical properties of the energy conversion and flux terms for steady, conservative motion. There is a certain lack of consistency in the classical as well as in a new proposed transformed Eulerian mean (TEM) energy budget for the individual conversion and flux terms. A nonzero divergence of the traditional time-mean energy flux vector, $\nabla \bar{\phi}$, does not necessarily indicate an increase of wave activity. An interdependence of the flux divergence and the conversion between the eddy and time-mean flows was noted by Dickinson (1969) and Hartmann (1976) and was demonstrated by Plumb (1983). Plumb also derived a new energy cycle using the TEM equations that eliminated the $p_a$ to $p_e$ conversion terms (where $P$ is the potential energy, $K$ is the kinetic energy, $z$ is a zonal mean, and $e$ is the eddy component). In fact, the energy flow for baroclinic waves was from $p_a$ to $K_e$ to $K_e$ and finally to $p_e$. This apparently surprising result can just as easily be interpreted in the context of the classical energy cycle where, by adding a constant to all the conversion terms, one can change the direction of the energy conversion in the traditional four box diagram (Orlanski, 1968). Take, for example, $[p_a \cdot p_e]$, the conversion from zonal potential energy to eddy potential energy. Substracting this term from the four conversion terms, $[p_a \cdot p_e], [p_a \cdot K_e], [K_e \cdot K_e], \text{and} [K_e \cdot p_e]$, results in an energy flow similar to the TEM approach proposed by Plumb (1983). The conversion between $p_a$ and $p_e$ is identically zero and the energy cycle will be from $p_a$ to $K_e$ to $K_e$ to $p_e$. Plumb’s main point was that all energy budgets have some ambiguities if partial terms are interpreted out of context; only the total contributions of the fluxes and conversions have a unique meaning.