Lecture 2  The Basic Conservation Laws

Control volumes + total differentiation

Control volume scale of molecules $\ll \delta V$ the scale of motion.

Eulerian framework: Control volume consists of a parallelepiped of sides $\delta x, \delta y, \delta z$, whose position is fixed related to the coordinates.

Lagrangian framework: control volume consists of an infinitesimal mass of “tagged” fluid particles; thus the control volume moves about following the motion of the fluid, always containing the same fluid particles.

The Lagrangian frame is particularly useful for deriving conservation laws, and the conservation of a quantity $Q$ can be expressed as $\frac{DQ}{Dt} = 0$. The Eulerian frame is, however, more convenient for solving most problems because in that system the field variables are related by a set of partial differential equations in which the independent variables are the coordinates $x, y, z$, and $t$.

The conservation contains expression for the rates of change of density, momentum, and thermodynamical energy following the motion of particular fluid particle (i.e., in Lagrangian frame). In order to apply these laws in the Eulerian frame it is necessary to derive a relationship between the rate of change of a field variable following the motion and its rate of change at a fixed point. The former is called the substantial, the total, or the material derivative (denoted by $\frac{D}{Dt}$). The latter is called the local derivative, merely the partial derivative w.r.t time.

These two frameworks are related by the concept of total differentiation.

Consider the temperature measured on a balloon that moves with the wind and suppose that this temperature is $T_0$ at the point $x_0, y_0, z_0$ and time $t_0$. If the balloon moves to the point $x_0 + \delta x, y_0 + \delta y, z_0 + \delta z$ in a time increment $\delta t$, then the temperature change recorded on the balloon, $\delta T$, can be expressed in a Taylor series expansion as

$$\delta T = \left( \frac{\partial T}{\partial t} \right) \delta t + \left( \frac{\partial T}{\partial x} \right) \delta x + \left( \frac{\partial T}{\partial y} \right) \delta y + \left( \frac{\partial T}{\partial z} \right) \delta z + \text{(higher order terms)}$$

The rate of change of $T$ following the motion becomes in the limit $\delta t \to 0$

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} \frac{Dx}{Dt} + \frac{\partial T}{\partial y} \frac{Dy}{Dt} + \frac{\partial T}{\partial z} \frac{Dz}{Dt}$$

If let

$$\frac{Dx}{Dt} \equiv u, \quad \frac{Dy}{Dt} \equiv v, \quad \frac{Dz}{Dt} \equiv w,$$

then, using vector notation,
\[
\frac{\partial T}{\partial t} = \frac{DT}{Dt} - U \cdot \nabla T
\]

where \( U = iu + jv + kw \) is the velocity vector. The 2\textsuperscript{nd} term on the rhs is called the temperature \textit{advection}.

\textbf{Rate of change of a vector in a rotating frame}
(Note the notation is slightly different from the other text.)

Consider first a vector of constant length relative to an inertial frame at a constant angular velocity \( \Omega \). Then, in a frame rotating with that same angular velocity it appears stationary and constant. If in a small interval of time \( \delta t \) the vector \( C \) rotates through a small angle \( \delta \lambda \) then the change in \( C \), as perceived in the inertial frame, is given by (see Fig. 2.1)

\[
\delta C = |C| \cos \vartheta \delta \lambda \mathbf{m}
\]

where the vector \( \mathbf{m} \) is the unit vector in the direction of change of \( C \), which is perpendicular to both \( C \) and \( \Omega \). But the rate of change of the angle \( \lambda \) is just, by definition, the angular velocity so that \( \delta \lambda = |\Omega| \delta t \) and

\[
\delta C = |C||\Omega| \sin \hat{\vartheta} \mathbf{m} \delta t = \Omega \times C \delta t
\]

using the definition of the vector cross product, where \( \hat{\vartheta} = (\pi / 2 - \vartheta) \) is the angle between \( \Omega \) and \( C \). Thus

\[
\left( \frac{DC}{Dt} \right) = \Omega \times C
\]

where the left-hand side is the rate of change of \( C \) as perceived in the inertial frame.
Now consider a vector $\mathbf{A}$ that changes in the inertial frame. In a small time $\delta t$ the change in $\mathbf{A}$ as seen in the rotating frame is related to the change seen in the inertial frame by

$$
(\delta \mathbf{A})_a = (\delta \mathbf{A})_R + (\delta \mathbf{A})_{\text{rot}}
$$

(2.4)

where the terms are, respectively, the change seen in the inertial frame, the change due to the vector itself changing as measured in the rotating frame, and the change due to the rotation. Using (2.2) and $(\delta \mathbf{A})_{\text{rot}} = \mathbf{\Omega} \times \mathbf{A} \delta t$ and so the rate of change of the vector $\mathbf{A}$ in the inertial and rotating frames are related by

$$
\left( \frac{D \mathbf{A}}{D t} \right)_a = \left( \frac{D \mathbf{A}}{D t} \right)_R + \mathbf{\Omega} \times \mathbf{A}
$$

(2.5)

This expression implies that viewing from the absolute (inertial) frame, the rotation of a vector should give rise to an extra rate of change than viewing from the rotating frame.

Note that for a scalar $B$, $\frac{D_a B}{D t} = \frac{D B}{D t}$.

**Conservation of momentum (Velocity and acceleration in a rotating frame)**

In an inertial frame, Newton’s law may be written as

$$
\frac{D_a U_a}{D t} = \sum_i F_i
$$

In Section 1.5 we found through simple physical reasoning that when the motion is viewed in a rotating coordinate system certain additional apparent forces must be included if Newton’s second low is to be valid. Here we show that the same results may be obtained by a formal transformation.

In order to transform the expression above to rotating coordinate, we must first find the relationship between $U_a$ and the velocity relative to the rotating system, which is denoted as $\mathbf{U}$. This relation is obtained by applying (2.5) to the position vector $\mathbf{r}$:

$$
\frac{D_a \mathbf{r}}{D t} = \frac{D \mathbf{r}}{D t} + \mathbf{\Omega} \times \mathbf{r}
$$

thus,

$$
U_a = U + \mathbf{\Omega} \times \mathbf{r}
$$

Now apply (2.5) to the velocity vector and obtain

$$
\frac{D_a U_a}{D t} = \frac{D U_a}{D t} + \mathbf{\Omega} \times U_a
$$

$$
= \frac{D}{D t} (U + \mathbf{\Omega} \times r) + \mathbf{\Omega} \times (U + \mathbf{\Omega} \times r)
$$

$$
= \frac{D U}{D t} + 2 \mathbf{\Omega} \times U - \mathbf{\Omega}^2 \mathbf{R}
$$

where $\mathbf{\Omega}$ is assumed to be constant and $\mathbf{R}$ is a vector perpendicular to the axis of rotation, with a magnitude equal to the distance to the axis of rotation.
If we assume that the real forces acting on the atmosphere are the pressure gradient force, gravitation, and friction, the Newton’s 2nd law can be written as

\[ \frac{DU}{Dt} = -2 \Omega \times U - \frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \tau \]

where the centrifugal force has been absorbed into the gravitation, the friction force is expressed as the gradient of the shearing stress.

**Spherical Coordinates:** \((\lambda, \phi, r)\)

Suppose coordinate direction unit vectors \(\mathbf{i}, \mathbf{j}, \mathbf{k}\) are pointing eastward, northward and upward, respectively, rotating uniformly with the Earth. Then, the components of velocity vector are

- \(u \equiv r \cos \phi \frac{D\lambda}{Dt} \)
- \(v \equiv r \frac{D\phi}{Dt} \)
- \(w \equiv \frac{Dz}{Dt} \) \( (r = a + z) \)

For notational simplicity, it is conventional to define \(x\) and \(y\) as eastward and northward distance, such that \(Dx = a \cos \phi D\lambda \) and \(Dy = rD\phi \). Thus the horizontal velocity components are \(u \equiv \frac{Dx}{Dt} \), \(v \equiv \frac{Dy}{Dt} \).

(Read Holton pages 31-37 to figure out yourself how the following set of equations on a spherical coordinate are derived, especially those curvature terms. Note the typos in page 32 and equation 2.13)

\[
\begin{align*}
\frac{Du}{Dt} - \frac{uv \tan \phi}{a} + \frac{uw}{a} &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + 2 \Omega v \sin \phi - 2 \Omega w \cos \phi + F_x \\
\frac{Dv}{Dt} + \frac{u^2 \tan \phi}{a} + \frac{vw}{a} &= -\frac{1}{\rho} \frac{\partial p}{\partial y} - 2 \Omega u \sin \phi + F_y \\
\frac{Dw}{Dt} - \frac{u^2 + v^2}{a} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + 2 \Omega u \cos \phi + F_z
\end{align*}
\]

The terms proportional to \(1/a\) on the left hand side are called curvature (metric) terms; they arise due to the curvature of the Earth surface. In addition to these nonlinear curvature terms, the terms hidden in the total derivatives also contain nonlinear terms:

\[
\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}
\]

This is why atmospheric dynamics is a very challenging subject, which makes it an interesting one too.

Some basic operators in spherical coordinate:
\[ \nabla \Phi = \frac{i}{r \cos \phi} \frac{\partial \Phi}{\partial \lambda} + \frac{j}{r} \frac{\partial \Phi}{\partial \phi} + k \frac{\partial \Phi}{\partial r} \]
\[ \nabla \cdot \mathbf{V} = \frac{1}{r \cos \phi} \left( \frac{\partial u}{\partial \lambda} + \frac{\partial (v \cos \phi)}{\partial \phi} \right) \]
\[ \mathbf{k} \cdot \nabla \times \mathbf{V} = \frac{1}{r \cos \phi} \left( \frac{\partial v}{\partial \lambda} - \frac{\partial (u \cos \phi)}{\partial \phi} \right) \]
\[ \nabla^2 \Phi = \frac{1}{r^2 \cos^2 \phi} \left[ \frac{\partial^2 \Phi}{\partial \lambda^2} + \cos \phi \frac{\partial}{\partial \phi} \left( \cos \phi \frac{\partial \Phi}{\partial \phi} \right) \right] \]

**Conservation of Mass—Continuity Equation**
Eulerian derivative (business as usual)

![Diagram](image)

**Fig. 2.5** Mass inflow into a fixed (Eulerian) control volume due to motion parallel to the x-axis.

mass flux rate per area in x-direction: \( \rho u \)
inflow of mass through the left-hand face: \( \rho u - \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \)
outflow of mass through the right-hand face: \( \rho u + \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \)

\[ \rightarrow \]
Net flow into box:
\[
\left[ \rho u - \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \right] \delta y \delta z - \left[ \rho u + \frac{\partial}{\partial x} (\rho u) \frac{\delta x}{2} \right] \delta y \delta z
\]

\[= -\frac{\partial}{\partial x} (\rho u) \delta x \delta y \delta z \]

Generalizing to 3-dimensional, the net rate of mass inflow is \(-\nabla \cdot \rho \mathbf{U}\) per unit volume, which is must be balanced by net accumulation rate \(\frac{\partial \rho}{\partial t}\), thus

\[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{U} = 0\]

This is the mass divergence form of the continuity equation.

Alternatively,

\[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{U} = \frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla \rho = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{U} = 0\]

So,

\[\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \mathbf{U} = 0\]

It states that the fractional rate of increase of the density following the motion of an air parcel is equal to minus the velocity divergence.

**Conservation of Mass following the fluid (Lagrangian Derivation)**

![Fig. 2.6](image)

**Fig. 2.6** Change in Lagrangian control volume (shown by shading) due to fluid motion parallel to the \(x\) axis.

Consider a “material volume” fluid enclosed by a “material surface” which is defined as a surface through which no mass flows. The mass \(\delta M\), of the control volume is

\[\delta M = \rho \delta x \delta y \delta z\]

where we assume the shape of the material surface to be parallelepiped. Because mass is conserved following the motion, we have
\[
\frac{1}{\delta M} D_{\Delta t}(\delta M) = \frac{1}{\rho \delta V} D_{\Delta t}(\rho \delta V) = \frac{1}{\rho} D_{\Delta t}(\rho) + \frac{1}{\delta V} D_{\Delta t}((\delta V) = 0
\]

while
\[
\frac{1}{\delta V} D_{\Delta t}(\delta V) = \frac{1}{\delta x} D_{\Delta t}(\delta x) + \frac{1}{\delta y} D_{\Delta t}(\delta y) + \frac{1}{\delta z} D_{\Delta t}(\delta z)
\]

\[
u_A = \frac{D}{D_t} \left( x_0 - \frac{\delta x}{2} \right)
\]

\[
u_B = \frac{D}{D_t} \left( x_0 + \frac{\delta x}{2} \right)
\]

\[
u_B - \nu_A = \frac{D}{D_t} \delta x = \delta u
\]

Likewise for \( v \) and \( w \).

\[
\lim_{\delta V \to 0} \frac{1}{\delta V} D_{\Delta t}(\delta V) = \lim_{\delta x \to 0} \frac{1}{\delta x} D_{\Delta t}(\delta x) + \lim_{\delta y \to 0} \frac{1}{\delta y} D_{\Delta t}(\delta y) + \lim_{\delta z \to 0} \frac{1}{\delta z} D_{\Delta t}(\delta z)
\]

\[
= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
\]

So
\[
\frac{1}{\rho} D_{\Delta t}(\rho) + \nabla \cdot U = 0 \quad (2.31)
\]

as before.

**Approximations related to continuity equation:**

*Incompressible:* \( \frac{D\rho}{D_t} = 0 \); or \( \nabla \cdot U = 0 \)

*Anelastic (weak Boussinesq) approximation:* \( \nabla \cdot \rho U = 0 \)

*Boussinesq approximation:* \( \nabla \cdot U = 0 \) The Boussinesq approximation ignores all variations of density of a fluid in the continuity and momentum equations, except when associated with the gravitational (or buoyancy) term. But this should absolutely not allow one to go back and use (2.31) to say that \( \frac{D\rho}{D_t} = 0 \); the evolution of density is given by the thermodynamic equation in conjunction with an equation of state.
CONSERVATION OF ENERGY—FIRST LAW OF THERMODYNAMICS

\[
\text{(CHANGE OF INTERNAL ENERGY)} = (\text{HEAT ADDED}) - (\text{WORK DONE by the system})
\]

\[
c_v \frac{DT}{Dt} = J - p \frac{D\alpha}{Dt}
\]

where \(c_v\) is the specific heat at constant volume.

This is a static statement, does it also hold for fluid in motion?

For fluid in motion, we must account for kinetic energy of the system:

\[
e = c_v T \quad \text{is internal energy per unit mass (due to temperature, i.e., molecular motion)}
\]

\[
K = (1/2) U \cdot U \quad \text{is kinetic energy}
\]

Therefore, total thermodynamic energy of (Lagrangian) control volume of fluid, \(\delta V\), is

\[
E = \left[ e + \frac{1}{2} U \cdot U \right] \rho \delta V
\]

To compute work done onto the control volume, take the product of resultant of external forces and velocity vector:

\[
W = \sum_i F_i \cdot U
\]

Recall that the forces are pressure force, gravity, Coriolis force, friction.

For the Coriolis force,

\[
W_{\text{Coriolis}} = (-2 \Omega \times U) \cdot U = 0
\]

Coriolis force is always perpendicular to the direction of motion and can do no work.

For the gravity,

\[
W_{\text{gravity}} = \rho g \cdot U \delta V = -\rho gw \delta V
\]

To derive the word done by PF (pressure force), consider parallelepiped control volume

\[
W_{PF}^{A,B} = \{(pu)_A - (pu)_B\} \delta y \delta z
\]

but \((pu)_B = (pu)_A + \left[ \frac{\partial (pu)}{\partial x} \right]_A \delta x + \cdots\)

so,

\[
W_{PF}^{A,B} = -\left[ \frac{\partial (pu)}{\partial x} \right]_A \delta x \delta y \delta z
\]

In general, the total rate at which work in done to the system by PF is

\[
W_{PF} = -\nabla \cdot (pU) \delta V
\]
Conservation of Energy

In summary,

\[
\frac{D}{Dt} \left[ \rho \left( e + \frac{1}{2} \mathbf{U} \cdot \mathbf{U} \right) \delta V \right] = -\nabla \cdot (\rho \mathbf{U}) \delta V + \rho \mathbf{g} \cdot \mathbf{U} \delta V + \rho J \delta V \tag{2.35}
\]

Here J is the rate of heating per unit mass due to radiation, conduction and latent heat release.

Expanding

\[
\rho \delta V \frac{D}{Dt} \left( e + \frac{1}{2} \mathbf{U} \cdot \mathbf{U} \right) + \left( e + \frac{1}{2} \mathbf{U} \cdot \mathbf{U} \right) \frac{D(\rho \delta V)}{Dt} = -\mathbf{U} \cdot \nabla p \delta V - \rho \nabla \cdot \mathbf{U} \delta V - \rho g \delta V + \rho J \delta V \tag{2.36}
\]

Recall the continuity equation gives \( \frac{D(\rho \delta V)}{Dt} = 0 \), thus

\[
\rho \frac{D}{Dt} \left( e + \frac{1}{2} \mathbf{U} \cdot \mathbf{U} \right) = -\mathbf{U} \cdot \nabla p - \rho \nabla \cdot \mathbf{U} - \rho g \delta V + \rho J \tag{2.37}
\]

This can be further simplified by taking \( \rho \mathbf{U} \) (momentum equation),

\[
\rho \frac{D}{Dt} \left( \frac{1}{2} \mathbf{U} \cdot \mathbf{U} \right) = -\mathbf{U} \cdot \nabla p - \rho g \delta V \quad \text{----mechanical energy equation} \tag{2.38}
\]

Subtracting (2.38) from (2.37), we obtain,

\[
\rho \frac{De}{Dt} = -p \nabla \cdot \mathbf{U} + \rho J \quad \text{----thermal energy equation} \tag{2.39}
\]

Using the definition of geopotential, \( g \delta V = g \frac{Dz}{Dt} = \frac{D \Phi}{Dt} \), we have

\[
\rho \frac{D}{Dt} \left( \Phi + \frac{1}{2} \mathbf{U} \cdot \mathbf{U} \right) = -\mathbf{U} \cdot \nabla p \tag{2.40}
\]
This is referred to as the mechanical energy equation. The sum of kinetic energy plus the gravitational potential energy is called the mechanical energy. Thus, following the motion, the rate of change of mechanical energy per unit volume equals the rate at which work is done by the pressure gradient force.

Noting that

\[-\frac{1}{\rho} \nabla \cdot \mathbf{U} = \frac{1}{\rho} \frac{D\rho}{Dt} = \frac{D\alpha}{Dt},\]

and that for dry air the internal energy per unit mass is \(e = c_v T\), where \(c_v = 717 \text{ J kg}^{-1} \text{ K}^{-1}\) is the specific heat at constant volume, we obtain the familiar form of thermodynamic energy equation from (2.39):

\[c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} = J \quad (2.41)\]

Thus the first law of thermodynamics indeed is applicable to a fluid in motion. The second term on the lfs, representing the rate of working by the fluid system, represents a conversion between thermal and mechanical energy. It is this conversion process enables the solar heat energy to drive the motion of the atmosphere.

**Thermodynamics of the dry atmosphere**

Recall the equation of state: \(P\alpha = RT\), and take total derivative,

\[p \frac{D\alpha}{Dt} + \alpha \frac{Dp}{Dt} = R \frac{DT}{Dt}\]

and substitute it into the thermodynamic energy equation, we obtain

\[c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J\]

Dividing through by \(T\),

\[c_p \frac{D\ln T}{Dt} - R \frac{D\ln p}{Dt} = J \frac{T}{T} \equiv \frac{Ds}{Dt}\]

This equation gives the rate of change of entropy per unit mass following the motion for a thermodynamically reversible process (the system has to be heated very slowly). A reversible process is one in which the system is in an equilibrium state throughout the process. Thus the system passes at an infinitesimal rate through a continuous succession of balanced states that are infinitesimally different from each other. In such a scenario, the process can be reversed, and the system and its environment will return to the initial state. For such a process, the entropy defined above is a field variable that depends only on the state of the fluid.

Note: Reversible does not necessarily mean adiabatic. A diabatic process could be reversible.

**Potential Temperature**

For an ideal gas undergoing an adiabatic process (reversible process in which no heat is exchanged with the surroundings), the first law gives
\[ c_p D \ln T - R D \ln p = 0 \]

So, we have a perfect differentials: \[ c_p D \ln T - R D \ln p \]

Integrate from \((p, T)\) to \((p_s, \theta)\):

\[ c_p \int_T^{\theta} D \ln T' = R \int_p^{p_s} D \ln p' , \text{ then } c_p \ln \left( \frac{\theta}{T} \right) = R \ln \left( \frac{p_s}{p} \right) . \]

\[ \theta = T \left( \frac{p_s}{p} \right)^{R/c_p} \text{ --Poisson’s equation} \]

**Potential temperature** is the temperature that a control volume of dry air initially at pressure \(p\) and temperature \(T\) would have if it were expanded or compressed adiabatically to a standard pressure \(p_s\). We usually take \(p_s\) to be 1000 hPa.

The way potential temperature is defined should lead immediately to

\[ c_p \frac{D \ln \theta}{Dt} = c_p \frac{D \ln T}{Dt} - R \frac{D \ln p}{Dt} = J \equiv \frac{Ds}{Dt} \]

Thus, for reversible processes, fractional potential temperature changes are indeed proportional to entropy changes. A parcel that conserves entropy must flow along an isentropic (constant \(\theta\)) surface.

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**Annual mean zonal mean temperature (°C)**

Annual mean zonal mean temperature (T) distribution
Annual mean zonal mean equivalent potential temperature ($\theta_e$) distribution
Adiabatic Lapse Rate

The lapse rate is the rate of temperature change with height. We have

\[ \ln \theta = \ln T + \frac{R}{c_p} \ln \frac{p_s}{p} = \ln T + \frac{R}{c_p} (\ln p_s - \ln p) \]

\[
\frac{d \ln \theta}{dz} = \frac{d \ln T}{dz} - \frac{R}{c_p} \frac{d \ln p}{dz}
\]

but

\[
\frac{dp}{dz} = -\rho g \Rightarrow \frac{d \ln p}{dz} = -\frac{\rho g}{p}
\]

so

\[
\frac{d \ln \theta}{dz} = \frac{d \ln T}{dz} + \frac{R \rho g}{c_p p}
\]

but \( \frac{p}{\rho} = RT \), so

\[
\frac{1}{\theta} \frac{d \theta}{dz} = \frac{1}{T} \frac{dT}{dz} + \frac{R}{c_p} \frac{g}{RT} = \frac{1}{T} \left( \frac{dT}{dz} + \frac{g}{c_p} \right)
\]

or

\[
\frac{T}{\theta} \frac{d \theta}{dz} = \frac{dT}{dz} + \frac{g}{c_p}
\]

If \( \frac{d \theta}{dz} = 0 \), i.e., potential temperature is constant w.r.t height,

\[
0 = \frac{dT}{dz} + \frac{g}{c_p} \Rightarrow \frac{dT}{dz} = \frac{g}{c_p} \equiv \Gamma_d
\]

\( \Gamma_d \) is the dry adiabatic lapse rate and approximately constant throughout the lower atmosphere, \( \Gamma_d = \frac{9.81}{1004} = 9.8 K km^{-1} \)

Static Stability

\[
\frac{T}{\theta} \frac{d \theta}{dz} = \frac{dT}{dz} + \frac{g}{c_p} = \Gamma_d - \Gamma
\]

If \( \Gamma < \Gamma_d \), \( \theta \) increases with height. An air parcel undergoes an adiabatic displacement from its equilibrium level will be positively buoyant when displaced downward and negatively buoyant displaced upward so that it will tend to return to its equilibrium level and the atmosphere is said to be statically stable or stably stratified.

A measure of static stability is

\[ N^2 = \frac{g}{\theta} \frac{d \theta}{dz} \]
\( N \) is also called *buoyancy frequency* or *Brunt-Vaisala frequency*. Its link to the frequency of buoyancy oscillation can be derived as the following.

Suppose a parcel of fluid in a hydrostatic environment is pushed away from its equilibrium position. We have

\[
\frac{dp_0}{dz} = -\rho_0 g
\]

for the environment. The vertical acceleration of the parcel, \( \frac{dw}{dt} \), may be written as

\[
\frac{Dw}{Dt} = \frac{D^2 \delta z}{D t^2} = -g - \frac{1}{\rho} \frac{\partial p}{\partial z}
\]

where \( p \) and \( \rho \) are the pressure and density of the parcel. In the parcel method it is assumed that the pressure of the parcel adjusts instantaneously to the environmental pressure during the displacement, so \( p = p_0 \). This condition must to be true if the parcel is to leave no disturbance to its environment. Thus with the aid of hydrostatic equation, we have

\[
\frac{D^2 \delta z}{D t^2} = -g - \frac{1}{\rho} \frac{\partial p_0}{\partial z} = -g + \frac{1}{\rho} \rho_0 g = g \left( \frac{\rho_0 - \rho}{\rho} \right)
\]

but \( p = \rho RT \) and \( p_0 = \rho_0 RT_0 = p \) so \( \rho_0 T_0 = \rho T \), so \( T_0 (\rho_0 - \rho) = \rho T - \rho T_0 \)

then

\[
\frac{\rho_0 - \rho}{\rho} = \frac{\rho T - \rho T_0}{\rho T_0} = \frac{T - T_0}{T_0}
\]

Also

\[
\theta = T\left( \frac{p_0}{p} \right)^{R/c_p} \quad \text{and} \quad \theta_0 = T_0\left( \frac{p_0}{p_0} \right)^{R/c_p} = T_0\left( \frac{p_0}{p} \right)^{R/c_p}
\]

so,

\[
\frac{T - T_0}{T_0} = \frac{\theta - \theta_0}{\theta_0}
\]

Thus
\[
\frac{D^2}{Dt^2} (\delta z) = g \frac{T - T_0}{T_0} = g \frac{\theta - \theta_0}{\theta_0}
\]

Now using Taylor expansion to approximate \( \theta_0 \) at \( \delta z \):

\[
\theta_0 (\delta z) = \theta_0 (0) + \frac{d\theta_0}{dz} \delta z
\]

If the parcel is initially at level \( z=0 \) where the potential temperature is \( \theta(0) \), then is displaced adiabatically to \( \delta z \). Because the potential temperature of the parcel \( \theta \) is conserved, thus the potential temperature of the parcel is \( \theta(\delta z) = \theta_0 (0) \) and hence

\[
\theta(\delta z) - \theta_0 (\delta z) = \theta_0 (0) - \theta_0 (\delta z) = - \frac{d\theta_0}{dz} \delta z
\]

Therefore,

\[
\frac{D^2}{Dt^2} (\delta z) = g \frac{\theta - \theta_0}{\theta_0} = - g \frac{d\theta_0}{dz} \delta z = - N^2 \delta z
\]

(2.52)

where \( N^2 = g \frac{d\theta_0}{dz} \). Equation (2.52) has a general solution of the form \( \delta z = A \exp(\pm i|N|t) \).

Therefore, if \( N^2 > 0 \), the parcel oscillates about its initial level with a frequency of \( N \). For averaged tropospheric stratification, \( N \approx 1.2 \times 10^{-1} \) s\(^{-1} \) so that the period of a buoyancy oscillation is about 8 min.

In the case of \( N = 0 \), examination of (2.52) indicates that no acceleration force will exist and the parcel will be in neutral equilibrium with at its new level. However, if \( N^2 < 0 \) (potential temperature decreases with height), the displacement will increase exponentially with time, thus statically unstable. We thus arrive at the familiar gravitational or static stability criteria for dry air:

\[
\begin{align*}
N^2 &> 0 \quad \text{statically stable} \\
N^2 &= 0 \quad \text{statically neutral} \\
N^2 &< 0 \quad \text{statically unstable}
\end{align*}
\]
$\theta_e$ is the temperature (T) the parcel would reach if all the water vapor in the parcel were to condense, releasing its latent heat, and then the parcel was brought adiabatically to surface pressure $p_s$.

$$\theta_e = \theta \exp \left( \frac{L_c q_s}{c_p T} \right)$$  \hspace{1cm} (9.40)

where $L_c$ is latent heat of condensation, $q_s$ is the saturation mixing ratio.
We know the dry adiabatic lapse rate $\Gamma_d = -9.8 K/km$ and the typical lapse rate of the troposphere $\Gamma = -\frac{dT}{dz} = 6.5 K/km$. Please estimate the frequency (or period) of the buoyancy oscillation in the typical troposphere. You may assume the temperature to be $T=280^\circ$ Kelvin.

(from Marshall and Plumb, 2008)