Lecture 6 Circulation and Vorticity

Given the rotation of the Earth, we are interested in the rotation of the atmosphere, but we have a problem: the atmosphere is a fluid, so the notion of “rotation” needs to be altered from our standard notions for the rotation of solid objects. The quantity that represents rotation in a fluid is called circulation, which involves the mathematical operation of a line integral. Separate notes are provided on the line integral.

![Diagram of circulation](image)

**Fig. 4.1 Circulation about a closed contour**

We define circulation as the line integral of the vector velocity

$$C = \oint_C \mathbf{V} \cdot d\mathbf{l}$$

where $C$ is the curve in space along which we integrate and $d\mathbf{l}$ is the infinitesimal element of distance along $C$ with direction tangent to $C$. In general, $\mathbf{V}$ is not directed along $d\mathbf{l}$, so it forms an angle, $\alpha$, w.r.t. $d\mathbf{l}$. Thus, unlike angular momentum or angular velocity, circulation can be computed without a reference to an axis of rotation; it can thus be used to characterize fluid rotation in situations where “angular velocity” is not defined easily.

How is this related to rotation? Consider a circular disk of fluid rotating about the z-axis at an angular velocity $\Omega$:

$$C = \oint_C (\Omega \times \mathbf{R}) \cdot d\mathbf{l}$$

$$= \oint_C |\Omega \times \mathbf{R}| \, dl$$

$$= \oint_C \Omega R \, dl$$ since $dl = R \, d\lambda$

$$= \int_0^{2\pi} \Omega R^2 \, d\lambda$$

$$= 2\pi \Omega R^2$$
so
\[ \frac{C}{\pi R^2} = 2\Omega \] circulation per unit area/vorticity

**Circulation theorem**

The circulation theorem is obtained by taking the line integral of Newton’s 2nd law for a closed chain of fluid particles. In the absolute coordinate system the result (neglecting viscous term) is

\[ \oint \frac{D_a U_a}{D_a t} \cdot d\mathbf{l} = -\oint \frac{\nabla p}{\rho} \cdot d\mathbf{l} - \oint \nabla \Phi \cdot d\mathbf{l} \] (4.1)

where the gravitational force \( \mathbf{g} \) is represented as the gradient of the geopotential \( \Phi \).

Recall that the Newton’s 2nd law in the absolute coordinate system is

\[ \frac{D_a U_a}{D_a t} = -\alpha \nabla p + \mathbf{g} \]

Applying the chain rule to the lhs of (4.1) gives

\[ \frac{D_a U_a}{D_a t} \cdot d\mathbf{l} = \frac{D_a}{D_a t} (U_a \cdot d\mathbf{l}) - U_a \cdot dU_a \] (4.2)

Substituting (4.2) into (4.1) and using the fact that the line integral about a closed loop of a perfect differential is 0, so that

\[ \oint \nabla \Phi \cdot d\mathbf{l} = \oint d\Phi = 0 \]

and noting that

\[ \oint U_a \cdot dU_a = \frac{1}{2} \oint (U_a \cdot U_a) = 0 \]

we obtain the circulation theorem:

\[ \frac{DC_a}{Dt} = \frac{D}{Dt} \oint U_a \cdot d\mathbf{l} = -\oint \frac{1}{\rho} \rho \cdot dp \] (4.3)

It is called *Bjerknes Circulation Theorem*.

The rhs of (4.3) is called the solenoidal term---a term from the Greek solenoëides, pipe-shaped. With the aid of Stoke’s theorem, we rewrite it as

\[ -\oint \frac{\nabla p}{\rho} \cdot d\mathbf{l} = -\iint_A \nabla \times \left( \frac{\nabla p}{\rho} \right) \cdot \mathbf{n} dA \]

\[ = \iint_A \left[ \nabla \times \left( \frac{1}{\rho} \right) \cdot \nabla p + \frac{1}{\rho} \nabla \times \nabla p \right] \cdot \mathbf{n} dA \]

\[ = \iint_A \left( \frac{\nabla p \times \nabla p}{\rho^2} \right) \cdot \mathbf{n} dA \]

(use the identities in the handout for vector calculus)

If the surfaces of constant pressure and constant density do not coincide, the fluid is termed *baroclinic*. If the fluid is baroclinic, then the baroclinic vector \( \nabla p \times \nabla p / \rho^2 \neq 0 \), and the circulation will change with time. An example is shown below.
Fig 2.2.4 (Pedlosky’s GFD book) The baroclinic production of vorticity. Surfaces of constant $p$ and $\rho$ which are not coincident will tend to increase the circulation as shown.

A barotropic fluid is one for which density is solely a function of pressure, so $\alpha = \alpha(p)$ and $\int \alpha \, dp = 0$. Therefore

$$\frac{DC_\alpha}{Dt} = 0$$

This is the Kelvin’s circulation theorem, stating that the absolute circulation is conserved following the motion for a barotropic fluid. It has important implication of finite-amplitude wave activity (or pseudo-momentum) budget on isentropic/isopycnal surfaces in the midlatitude atmospheric/ocean dynamics.
Relative Circulation

We have \( C_a \equiv \oint U_a \cdot d\mathbf{l} \), where \( U \) is relative velocity with respect to rotating earth. 

\[ U_a = U + \Omega \times r \]

Then

\[ C_a = \oint U \cdot d\mathbf{l} + \oint (\Omega \times r) \cdot d\mathbf{l} \]

\[ = C_r + \oint (\Omega \times r) \cdot d\mathbf{l} \]

The 2\(^{nd}\) term on the rhs becomes, using the Stoke’s theorem again,

\[ \oint (\Omega \times r) \cdot d\mathbf{l} = \iint_A \nabla \times (\Omega \times r) \cdot \mathbf{n} \\ dA \]

while \( \nabla \times (\Omega \times r) = \Omega \nabla \cdot \mathbf{R} = 2\Omega \)

so

\[ \oint (\Omega \times r) \cdot d\mathbf{l} = \iint_A 2\Omega \cdot \mathbf{n} \\ dA = 2\Omega A_e \]

![Diagram](image)

**Fig. 4.2** Area \( A_e \) subtended on the equatorial plane by horizontal area \( A \) centered at latitude \( \phi \).

Combining, we have

\[ \frac{DC_a}{Dt} = \frac{DC_r}{Dt} + 2\Omega \frac{DA_e}{Dt} \]

so

\[ \frac{DC_r}{Dt} = -\oint \alpha dp - 2\Omega \frac{DA_e}{Dt} \]

**Bjerknes circulation theorem (relative)**

where \( A_e \) can change because

(i) area of domain
(ii) meridional location
(iii) inclination of domain

all can vary.
For a barotropic fluid, \( \alpha = \alpha(p) \) so

\[
\frac{DC_r}{Dt} = -2\Omega \frac{DA_r}{Dt}
\]
or

\[
C_r(t_2) - C_r(t_1) = -2\Omega \left[ A_r(t_2) - A_r(t_1) \right]
\]

**Example:** Suppose that air within a circular region of radius 100km centered at the equator is initially motionless with respect to the earth. If this circular air mass were moved to the North Pole along an isobaric surface preserving its area, the circulation about the circumference would be

\[
C = -2\Omega \pi r^2 \left[ \sin(\pi / 2) - \sin(0) \right]
\]

Thus the mean tangential velocity at the radius would be

\[
V = C / (2\pi r) = -\Omega r = -7 \text{ m s}^{-1}
\]
The negative sign indicates the air has acquired anticyclonic relative circulation.

**Absolute Vorticity**
We define the pointwise measure of rotation in a fluid as its *vorticity*, given by the curl of the velocity:

\[
\omega_a = \nabla \times U_a
\]

In large-scale dynamics, we are usually only concerned with the vertical component of the vorticity---or horizontal rotation.

\[
\eta \equiv \mathbf{k} \cdot \omega_a = \mathbf{k} \cdot (\nabla \times U_a)
\]

Taking the Stoke’s form of circulation:

\[
C_a = \oint U_a \cdot d\mathbf{l} = \iint_A \omega_a \cdot n \, dA
\]

Thus, Stoke’s theorem states that the circulation about any close loop is equal to the integral of the normal component of vorticity over the area enclosed by the contour. As a consequence, the vorticity of a fluid in solid body rotation is just twice the angular velocity. *Vorticity* may thus be regarded as a measure of the local \((A \to 0)\) angular velocity of the fluid.

For a circuit in the horizontal plane with an area of \(A\), the vorticity may be also defined as

\[
\eta \equiv \lim_{A \to 0} \frac{C_a}{A}
\]

where \(A\) is the area enclosed by the circuit.

**Relative Vorticity**
As above:

\[
\omega_r = \nabla \times U_r
\]

In Cartesian coordinate,
\[ \omega_r = \left( \frac{\partial v}{\partial y} - \frac{\partial v}{\partial z} \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \frac{\partial v}{\partial x} - \frac{\partial w}{\partial y} \right) \]

(Use determinant to show it in Cartesian coordinate)

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\partial u & \partial v & \partial w \\
\end{vmatrix}
\]

And the relative vorticity in the vertical direction is:

\[ \zeta \equiv \mathbf{k} \cdot \omega_r = \mathbf{k} \cdot (\nabla \times \mathbf{U}_r) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]

or

\[ \zeta \equiv \lim_{A \to 0} \frac{C}{A} \]

The equivalence between these two definitions of the relative vorticity is illustrated in Fig. 4.4

Evaluating \( \mathbf{V} \cdot d\mathbf{l} \) for each side of the rectangle in Fig. 4.4 yields the circulation

\[
\delta C = u \delta x + \left( v + \frac{\partial v}{\partial x} \right) \delta y - \left( u + \frac{\partial u}{\partial y} \right) \delta x - v \delta y
\]

\[
= \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta x \delta y
\]

Dividing through by the area \( \delta A = \delta x \delta y \) gives
The difference between absolute and relative vorticity is *planetary vorticity*, which is just
the local vertical component of the vorticity of the earth due to its rotation:
\[ \mathbf{k} \cdot \nabla \times \mathbf{U} = \mathbf{k} \cdot 2\Omega = 2\Omega \sin \phi = f. \] Thus \( \eta = \zeta + f \).

**Potential Vorticity—General Derivation**

Although Kelvin’s circulation theorem is a general statement about vorticity
conservation, in its original form it is not a very useful statement for two reasons: 1. it
only applies to barotropic flow; 2. it is not a statement about a field, such as vorticity
itself. It turns out it is possible to derive a beautiful conservation law that overcomes both
of these failings and one, that is extraordinarily useful in geophysical fluid dynamics.
This is the conservation of potential vorticity (PV) introduced first by Rossby and then in
a more general form by Ertel. The idea is that we can use a scalar field that is being
advected by the flow to keep track of, or to take care of, the evolution of fluid elements.
For a baroclinic fluid this scalar field must be chosen in a special way, but there is no
restriction to a barotropic fluid. Then using the scalar evolution equation in conjunction
with the vorticity equation gives us a scalar conservation equation.

**Barotropic fluids**

![Fig. 4.8 An infinitesimal fluid element, bounded by two isosurfaces of the conserved tracer \( \chi \). As \( D\chi / Dt = 0 \), then \( D\delta\chi / Dt = 0 \).](image)

For an infinitesimal Lagrangian control volume we write Kelvin’s theorem as
\[ \frac{D}{Dt} [(\omega, \mathbf{n})\delta A] = 0 \quad (A6.1) \]

Now consider a volume bounded by two isosurface of values \( \chi \) and \( \chi + \delta\chi \), where \( \chi \)
is any materially conserved tracer, thus satisfying \( D\chi / Dt = 0 \), so that \( \delta A \) initially lies
in an isosurface of \( \chi \) (see figure below). \( \mathbf{n} \) is a unit vector normal to an infinitesimal
surface \( \delta A \). Thus \( \mathbf{n} = \nabla \chi / |\nabla \chi| \) and the infinitesimal volume \( \delta V = \delta h \delta A \). We have
\[(\omega \cdot n)\delta A = \omega \cdot \frac{\nabla \chi}{|\nabla \chi|} \frac{\delta V}{\delta h}\]

Now, \(\delta \chi = \delta x \cdot \nabla \chi = \delta h |\nabla \chi|\).

Using this in Kelvin’s Theorem (A6.1), we obtain
\[
\frac{D}{Dt} \left[ \frac{(\omega \cdot \nabla \chi)\delta V}{\delta \chi} \right] = 0
\]

Since \(\chi\) is conserved and the mass of the control volume is also conserved. So that this equation becomes
\[
\frac{\rho \delta V}{\delta \chi} \frac{D}{Dt} \left[ \frac{\omega}{\rho} \cdot \nabla \chi \right] = 0
\]

This equation is a statement of potential vorticity conservation for a barotropic fluid. The field \(\chi\) may be chosen arbitrarily, provided that it is materially conserved.

**The general case**

For a baroclinic fluid the above derivation fails simply because the statement of the conservation of circulation (A6.1) is not, in general, true: there are solenoidal terms on the rhs, i.e.,
\[
\frac{D}{Dt} [(\omega \cdot n)\delta A] = S_0 \cdot n \delta A, \quad S_0 = -\nabla \times \nabla p = -\nabla s \times \nabla T \tag{A6.2}
\]

The solenoidal term may be annihilated by choosing the circuit around which we evaluate the circulation to be such that the solenoidal term is identically zero. The choice of the circuit has to be along a surface of conserved quantity, this restricts the choice of \(\chi\) to be entropy \(s\) (or potential temperature \(\theta\)), or to be \(\rho\) density if the thermodynamic equation is \(\frac{D\rho}{Dt} = 0\). Choosing \(\chi\) to be \(\theta\), one can show the rhs
\[
S_0 \cdot n = 0
\]

\[
S_0 \cdot n = S_0 \cdot \frac{\nabla \theta}{|\nabla \theta|} = (-\nabla \times \nabla T) \cdot \frac{\nabla \theta}{|\nabla \theta|} = 0
\]

Thus, the conservation of potential vorticity for inviscid, adiabatic flow is
\[
\frac{D}{Dt} \left[ \frac{\omega}{\rho} \cdot \nabla \theta \right] = 0 \tag{A.6.3},
\]

where \(\frac{D\theta}{Dt} = 0\). A summary of the derivation of (A6.3) is provided by figure below.
Fig. 4.9 Geometry of potential vorticity conservation. The circulation equation is
\[ \frac{\partial (\omega_a \cdot n) \delta A}{\partial t} = S_o \cdot n \delta A, \]
where \( S_o \approx \nabla \theta \times \nabla T \). We choose \( n = \nabla \theta / |\nabla \theta| \),
where \( \theta \) is materially conserved, to annihilate the solenoidal term on the right-hand
side, and we note that \( \delta A = \delta V / \delta h \), where \( \delta V \) is the volume of the cylinder, and
the height of the column is \( \delta h = \delta \theta / |\nabla \theta| \). The circulation is
\[ C \equiv \omega_a \cdot n \delta A = \omega_a \cdot (\nabla \theta / |\nabla \theta|)(\delta V / \delta h) = [\rho^{-1} \omega_a \cdot \nabla \theta](\delta M / \delta \theta), \]
where \( \delta M = \rho \delta V \) is the mass of the cylinder. As \( \delta M \) and \( \delta \theta \) are materially conserved, so is the potential vorticity
\[ \rho^{-1} \omega_a \cdot \nabla \theta. \]
**Potential Vorticity—Isentropic Derivation**

Recall

\[ p = \rho RT \quad \theta = T \left( \frac{p_0}{p} \right)^{R/C_p} \quad C_p = C_v + R \]

\[ \rho = \frac{P}{RT} = \frac{P}{R} \left[ \theta \left( \frac{p_0}{p} \right)^{-R/C_p} \right]^{-1} = R^{-1} \theta^{-1} p^{-1} R^{-1/C_p} p_0^{R/C_p} \]

\[ \rho = R^{-1} \theta^{-1} p^{C_r/C_p} p_0^{\kappa} \quad \kappa = R / C_p \]

For an isentropic process, \( \frac{D}{Dt} \theta = 0 \), or \( \theta = \theta_0 \), so

\[ \rho \approx p^{C_r/C_p} \]

so that the solevoidal term becomes

\[ -\int_c \alpha dp \approx -\int_c p^{-C_v/C_p} dp = -\int_c dp^{C_v/C_p} = 0 \]

Thus, an adiabatic process has a simple circulation theorem:

\[ \frac{D}{Dt} \left( C_r + 2\Omega \sin \phi \delta A \right) = 0 \]

where we have used the Earth’s spherical geometry to express the area change.

Using the definition of relative vorticity, we write

\[ \frac{D}{Dt} \left[ \delta A (\zeta_\theta + f) \right] = 0 \quad f = 2\Omega \sin \phi \quad (4.11) \]

where \( \zeta_\theta \) designates the vertical component of relative vorticity evaluated on an isentropic surface. This conservation is only approximately true when the isentropic surfaces are approximately horizontal. Or the isentropes should slope very gently with horizontal position.

Now consider a cylindrical element of fluid contained between two isentropes, \( \theta_0 \) and \( \theta_0 + \delta \theta \). The mass of the fluid in the cylinder is

\[ \delta M = (-\delta p / g) \delta A \]

must be conserved following the motion. Therefore,

\[ \delta A = \frac{\delta M g}{\delta p} = \left( -\frac{\delta \theta}{\delta p} \right) \left( \frac{\delta M g}{\delta \theta} \right) = \text{const} \times g \left( -\frac{\delta \theta}{\delta p} \right) \]

as both \( \delta M \) and \( \delta \theta \) are constants. Substituting into (4.11) and taking the limit \( \delta p \to 0 \), we obtain

\[ \frac{D}{Dt} \left[ (\zeta_\theta + f) \left( -g \frac{\partial \theta}{\partial p} \right) \right] = 0 \quad (4.12) \]

The quantity in the square bracket is the isentropic *Ertel PV*.

In essence, PV is always in some sense a measure of the ratio of the absolute vorticity to the effective depth of the vortex. In (4.12), for example, the effective depth is just the
differential distance between potential temperature surfaces measured in pressure units \((-\partial \theta / \partial p)\). For a homogeneous incompressible fluid, PV conservation takes a somewhat simpler form:

\[
\frac{D}{Dt} \left[ \frac{(\zeta + f)}{h} \right] = 0
\]

(could be a good exam problem)

**Fig. 4.7** A cylindrical column of air moving adiabatically, conserving potential vorticity.

Example:

1) \(\frac{\partial \theta}{\partial p}\) = constant, \(\eta = \zeta + f\)

\[\Rightarrow \frac{d\eta}{dt} = 0 \quad \text{or} \quad \frac{d\zeta}{dt} = -\frac{df}{dt}\]

Suppose zonal flow at \((x_0, y_0)\), \(\zeta(x_0, y_0) = 0\), \(\eta(x_0, y_0) = f_0\)

Since any parcel that passes through \((x_0, y_0)\) must conserve \(\eta = \zeta + f = f_0\). In NH, \(f\) increases to north, so

\[y > y_0, \quad \zeta = f_0 - f < 0\]
\[y < y_0, \quad \zeta = f_0 - f > 0\]

2) if \(\frac{\partial \theta}{\partial p}\) (or the effective depth of the fluid) changes following the motion.

Westerly impinging upon an infinitely long mountain barrier.
3) Easterly impinging upon an infinitely long mountain barrier.

Fig. 4.10  As in Fig. 4.9, but for easterly flow.
Is this scenario possible?