RELATIVE VORTICITY EQUATION

Newton's law in a rotating frame in z-coordinate (frictionless):

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U}\cdot\nabla\mathbf{U} = -2\mathbf{\Omega}\times\mathbf{U} - \nabla\Phi - \alpha\nabla p$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla\left(\frac{\mathbf{U}\cdot\mathbf{U}}{2}\right) + (\nabla\times\mathbf{U})\times\mathbf{U} = -2\mathbf{\Omega}\times\mathbf{U} - \nabla\Phi - \alpha\nabla p$$

Applying $\nabla\times$ to both sides, and noting $\omega \equiv \nabla\times\mathbf{U}$ and using identities (the underlying tilde indicates vector):

$$\underset{\sim}{A}\cdot\underset{\sim}{\nabla}\underset{\sim}{A} = \frac{1}{2}\underset{\sim}{\nabla}(\underset{\sim}{A}\cdot\underset{\sim}{A}) + (\underset{\sim}{\nabla}\times\underset{\sim}{A})\times\underset{\sim}{A}$$

$$\underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\times\underset{\sim}{A}) = 0$$

$$\underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\gamma = 0$$

$$\underset{\sim}{\nabla}\times(\gamma\underset{\sim}{A}) = \underset{\sim}{\nabla}\gamma\times\underset{\sim}{A} + \gamma\,\underset{\sim}{\nabla}\times\underset{\sim}{A}$$

$$\underset{\sim}{\nabla}\times(\underset{\sim}{F}\times\underset{\sim}{G}) = \underset{\sim}{F}(\underset{\sim}{\nabla}\cdot\underset{\sim}{G}) - \underset{\sim}{G}(\underset{\sim}{\nabla}\cdot\underset{\sim}{F}) + (\underset{\sim}{G}\cdot\underset{\sim}{\nabla})\underset{\sim}{F} - (\underset{\sim}{F}\cdot\underset{\sim}{\nabla})\underset{\sim}{G}$$

So,

$$\frac{\partial \omega}{\partial t} + \nabla\times\left[\nabla\left(\frac{\mathbf{U}\cdot\mathbf{U}}{2}\right)\right] + \nabla\times(\omega\times\mathbf{U}) = -\nabla\times(2\mathbf{\Omega}\times\mathbf{U}) - \nabla\times\nabla\Phi - \nabla\times(\alpha\nabla p)$$

$$\Downarrow$$

$$\frac{\partial \omega}{\partial t} + \nabla\times(\omega\times\mathbf{U}) = -\nabla\times(2\mathbf{\Omega}\times\mathbf{U}) - \nabla\alpha\times\nabla p - \alpha\nabla\times\nabla p$$

Using $\mathbf{S}$ to denote baroclinicity vector, $\mathbf{S} = -\nabla\alpha\times\nabla p$, then,

$$\frac{\partial \omega}{\partial t} + \omega(\nabla\cdot\mathbf{U}) - \mathbf{U}(\nabla\cdot\omega) + (\mathbf{U}\cdot\nabla)\omega - (\omega\cdot\nabla)\mathbf{U} = -2\mathbf{\Omega}(\nabla\cdot\mathbf{U}) + \mathbf{U}\nabla\cdot(2\mathbf{\Omega}) - (\mathbf{U}\cdot\nabla)(2\mathbf{\Omega}) + (2\mathbf{\Omega}\cdot\nabla)\mathbf{U} + \mathbf{S}$$

$$\frac{\partial \omega}{\partial t} + \omega(\nabla\cdot\mathbf{U}) + (\mathbf{U}\cdot\nabla)\omega - (\omega\cdot\nabla)\mathbf{U} = -2\mathbf{\Omega}(\nabla\cdot\mathbf{U}) - (\mathbf{U}\cdot\nabla)(2\mathbf{\Omega}) + (2\mathbf{\Omega}\cdot\nabla)\mathbf{U} + \mathbf{S}$$

A little rearrangement:

$$\underbrace{\frac{\partial \omega}{\partial t}}_{\text{tendency}} = \underbrace{-(\mathbf{U}\cdot\nabla)(\omega+2\mathbf{\Omega})}_{\text{advection}} \underbrace{-(\omega+2\mathbf{\Omega})(\nabla\cdot\mathbf{U})}_{\text{convergence}} + \underbrace{\left[(\omega+2\mathbf{\Omega})\cdot\nabla\right]\mathbf{U}}_{\text{twisting}} + \underbrace{\mathbf{S}}_{\text{baroclinicity}}$$

RELATIVE VORTICITY EQUATION (Vertical Component)

Take $\mathbf{k}\cdot$ on the 3-D vorticity equation:

$$\frac{\partial \zeta}{\partial t} = -\mathbf{k}\cdot(\mathbf{U}\cdot\nabla)(\omega+2\mathbf{\Omega}) - \mathbf{k}\cdot(\omega+2\mathbf{\Omega})(\nabla\cdot\mathbf{U}) + \mathbf{k}\cdot\left[(\omega+2\mathbf{\Omega})\cdot\nabla\right]\mathbf{U} + \mathbf{k}\cdot\mathbf{S}$$

In isobaric coordinates, $\mathbf{k} = \frac{\nabla p}{|\nabla p|}$, so $\mathbf{k}\cdot\mathbf{S} = 0$. Working through all the dot product, you should get vorticity equation for the vertical component as shown in (A7.1).

The same vorticity equations can be easily derived by applying the curl to the horizontal momentum equations (in p-coordinates, for example).

$$-\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p} = \frac{uv\tan\phi}{a} + \frac{u\omega}{\rho g a} + \frac{f'\omega}{\rho g} + fv - \frac{\partial \Phi}{\partial x}\right]$$
$$+\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial p} = -\frac{u^2\tan\phi}{a} + \frac{v\omega}{\rho g a} - fu - \frac{\partial \Phi}{\partial y}\right]$$
$$\Downarrow$$

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \omega\frac{\partial \zeta}{\partial p} + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}\right)$$
$$= -v\frac{\partial f}{\partial y} - u\frac{\partial f}{\partial x} + \text{metric terms} + \cdots$$

expressed in a vectorial form with the metric terms neglected:

$$\frac{\partial \zeta}{\partial t} = -\mathbf{V}\cdot\nabla(\zeta + f) - (\zeta + f)\nabla_h\cdot\mathbf{V} - \left(\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}\right) \quad + \quad 0 \qquad \text{(A7.1)}$$

tendency advection convergence twisting/tilting no baroclinicity term

Physical meaning of the terms in vorticity equation

Tendency: local changes; think of this as the accounting sheet.

Advection: importation (flux) of vorticity from elsewhere

β-effet: $-v\frac{\partial f}{\partial y} = -\beta v$; the restoring force due to meridional variation of Coriolis parameter --- important in the planetary waves, mountain lee waves and equatorial Rossby waves.

Convergence: change in the length of the moment arm---solid body analogue

Twisting:

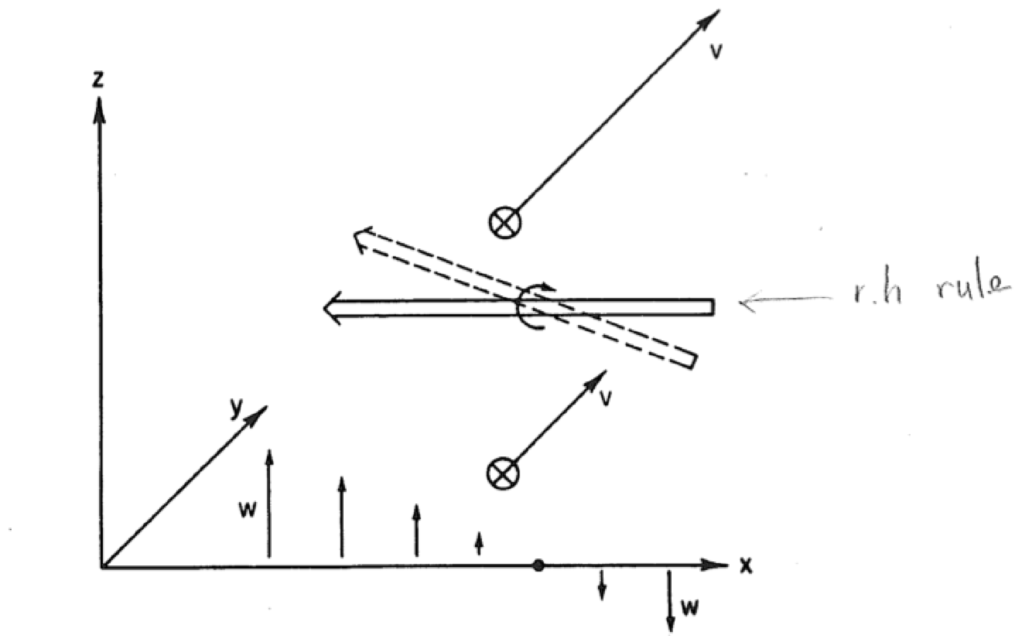

Fig. 4.12 Vorticity generation by the tilting of a horizontal vorticity vector (double arrow).

<u>Scale analysis---Continuity equation</u>

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0$$

$u,v \sim U \qquad \sim 10 ms^{-1}$

$w \sim W \qquad \sim 10^{-2} ms^{-1}$

$x,y \sim L \qquad \sim 10^{6} m$

$\frac{\partial \omega}{\partial p} \sim \frac{\omega}{\Delta p} \sim \frac{\omega}{\rho g \Delta H} \sim \frac{W}{\Delta H} \quad bc. \;\; W \sim \frac{\omega}{\rho g}$

$\Delta H \sim 10^{4} m$

$\frac{\partial \omega}{\partial p} \sim 10^{-6} ms^{-2}$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0$$

$$\frac{U}{L} \quad \frac{U}{L} \quad \frac{W}{H}$$

$$10^{-5} \quad 10^{-5} \quad 10^{-6}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=O\left(\frac{\partial \omega}{\partial p}\right)\ll O\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\ll\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

or

$$D\ll\zeta$$

Scale analysis---relative vorticity equation

For typical synoptic scale,

$u,v\sim U \quad \sim 10 ms^{-1}$

$w\sim W \quad \sim 10^{-2} ms^{-1}$

$x,y\sim L \quad \sim 10^{6} m$

$H\sim 10^{4} m$

$(\Delta p)_h \sim 10\ hPa$

$\rho_0 \sim 1\ Kg\ m^{-3}$

$\Delta\rho/\rho_0 \sim 10^{-2}$

$f_0 \sim 10^{-4} s^{-1}$

so

$\frac{L}{U}\sim 10^{5} s$

$\zeta\sim\frac{U}{L}\sim 10^{-5} s^{-1}$

while $\frac{\partial f}{\partial y}\equiv\beta\sim 10^{-11}\ m^{-1}s^{-1}$

so

$\frac{\zeta}{f_0}\sim\frac{U}{f_0 L}\sim 10^{-1}$ (Rossby number)

$$\frac{\partial \zeta}{\partial t} = -u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} - \omega\frac{\partial \zeta}{\partial p} - v\frac{\partial f}{\partial y} - (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}\right)$$

$$\frac{U}{LT} \quad \frac{U^2}{L^2} \quad \frac{U^2}{L^2} \quad \frac{UW}{LH} \quad U\beta \quad \frac{U}{L}, f_0 \quad \frac{W}{H} \quad \frac{UW}{LH} \quad \frac{UW}{LH}$$

$$10^{-10} \quad 10^{-10} \quad 10^{-10} \quad 10^{-11} \quad 10^{-10} \quad 10^{-11}, 10^{-10} \quad 10^{-11} \quad 10^{-11}$$

Therefore,

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \beta v = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

but

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

So

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} + \beta v = f\frac{\partial \omega}{\partial p} \tag{4.22a}$$

Note that (4.22a) is not accurate in intense cyclonic storms. For these the relative vorticity should be retained in the divergence term:

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.22b}$$

The form of vorticity equation given in (4.22b) also reveals why cyclonic disturbances can be much more intense than anticyclones. Provided that the magnitude of convergence is fixed, relative vorticity will increase, and the factor $(\zeta + f)$ becomes larger, which leads to even higher rate of increase in the relative vorticity. For a fixed rate of divergence, however, relative vorticity will decrease, which will decrease $(\zeta + f)$. When $\zeta \to -f$, the divergence term on the right hand side approaches zero and the relative vorticity cannot become more negative no matter how strong the divergence.

The approximate forms given in (4.22a) and (4.22b) do not remain valid, however, in the vicinity of atmospheric fronts. The horizontal scale of variation in frontal zones is only ~100km, and the vertical velocity scale is ~10cm/s. For these scales, vertical advection, tilting, and solenoidal terms all may become as large as the divergence term.

For the homogeneous, incompressible fluid, the vorticity equation can be written as

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

in z-coordinate.

$$\downarrow$$

$$\frac{D_h}{Dt}\left(\frac{\zeta+f}{H}\right)=0$$

Because

$$\frac{1}{A}\frac{dA}{dt}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

and volume V is conserved.

$$\frac{D_h\left(\zeta+f\right)}{Dt}=-\left(\zeta+f\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\downarrow$$

$$\frac{D_h\left(\zeta+f\right)}{Dt}=-\left(\zeta+f\right)\left(\frac{1}{A}\frac{DA}{Dt}\right)$$

$$V=AH;\;\; DV/Dt=0\Rightarrow\frac{1}{A}\frac{dA}{dt}=-\frac{1}{H}\frac{dH}{dt}\qquad\downarrow$$

$$\frac{D_h\left(\zeta+f\right)}{Dt}=\left(\zeta+f\right)\left(\frac{1}{H}\frac{DH}{Dt}\right)$$

$$\downarrow$$

$$\frac{D_h}{Dt}\left(\frac{\zeta+f}{H}\right)=0$$

IMPORTANT CONCEPTS

1. Circulation is mathematical way to represent rotation in a fluid without knowing anything about the axis of rotation.
2. Bjerknes: Absolute circulation is changed by solenoidal term
3. Kelvin: Absolute circulation is conserved in barotropic fluid
4. Relative vorticity can be changed by solenoid, area of domain, latitude, or inclination.
5. Vorticity is the circulation per area, i.e., pointwise quantity representing rotation
6. Potential vorticity --- ratio of absolute vorticity to effective depth is conserved under the conditions of adiabatic, inviscid flow.
7. Conservation of total/absolute vorticity gives rise to Rossby wave whose restoring force is the meridional gradient of the planetary vorticity (f).
8. Vorticity can change locally due to advection, convergence (area), twisting from one component to another, or baroclinicity (solenoid).
9. The insights and understanding of dynamics from the shallow water potential vorticity can be readily carried over to the PV dynamics on an isentropic surface.

Potential Vorticity in isentropic coordinates—Ertel PV

For a control volume in θ coordinate, $\delta M = \sigma \delta A \delta\theta$, where $\sigma \equiv -\frac{1}{g}\frac{\partial p}{\partial \theta}$, has the role of "density" in θ coordinate. The vertical component of the relative vorticity is

$$\zeta_\theta \equiv \mathbf{k} \cdot \nabla_\theta \times \mathbf{V}$$

The Ertel PV is defined as

$$P \equiv \left(\zeta_\theta + f\right)\left(-g\partial\theta / \partial p\right) = \frac{\zeta_\theta + f}{\sigma}$$

Weather disturbances that have sharp gradients in dynamical fields, such as jets and fronts, are associated with large anomalies in the Ertel PV. In the upper troposphere such anomalies tend to be advected rapidly under nearly adiabatic conditions. Thus, the potential vorticity anomaly patterns are conserved materially on isentropic surfaces. This material conservation property makes Ertel PV anomalies particularly useful in identifying and tracing the evolution of meteorological disturbances.

In Isentropic coordinates, one can also define

Montgomery streamfunction: $\Psi \equiv c_p T + \Phi$

and

Exner function: $\Pi \equiv c_p \left(\frac{p}{p_s}\right)^{R/C_p} = c_p \frac{T}{\theta}$