DIAGNOSIS OF THE VERTICAL MOTION

1. vorticity equation

\[
\frac{\partial \zeta}{\partial t} = -V_g \cdot \nabla (\zeta + f) + f_0 \frac{\partial \omega}{\partial p}
\] (6.19)

2. Continuity equation (Section 3.5.1)

3. Omega equation:

\[
\left( \nabla^2 + \frac{f_0^2}{\sigma} \frac{\partial^2}{\partial p^2} \right) \omega = \frac{f_0}{\sigma} \frac{\partial}{\partial p} \left[ V_g \cdot \nabla \left( \frac{1}{f_0} \nabla^2 \Phi + f \right) \right]
\]

\[
+ \frac{1}{\sigma} \nabla^2 \left[ V_g \cdot \nabla \left( \frac{\partial \Phi}{\partial p} \right) \right] - \frac{\kappa}{\sigma p} \nabla^2 J
\]

(6.34)

Because of large cancellation between B and C terms, the better form of the omega diagnostic equation for adiabatic flow is

\[
\left( \nabla^2 + \frac{f_0^2}{\sigma} \frac{\partial^2}{\partial p^2} \right) \omega \approx \frac{f_0}{\sigma} \frac{\partial}{\partial p} \left[ V_g \cdot \nabla \left( \frac{1}{f_0} \nabla^2 \Phi + f \right) \right]
\]

(6.36)

For qualitative discussion it is permissible to assume that \( \omega \) has a sinusoidal structure in vertical:

\[
\omega = W_0 \sin(\pi p / p_0) \sin kx \sin ly
\]

(6.37)

we then can write the lfs of (6.36) as

\[
\left( \nabla^2 + \frac{f_0^2}{\sigma} \frac{\partial^2}{\partial p^2} \right) \omega \approx - \left[ k^2 + l^2 + \frac{1}{\sigma} \left( \frac{f_0 \pi}{p_0} \right)^2 \right] \omega
\]

The rhs of (6.36) represents the advection of absolute vorticity by the thermal wind. For a typical developing baroclinic system (eastward propagating), the geopotential, isotherms and vertical motion at 500 hPa level are indicated in Figure 6.12. Due to the westward tilt of the system with height, the 500-hPa geopotential leads the isotherm pattern.

\[
\frac{\partial V_g}{\partial z} \cdot \nabla (\zeta + f) \left\{ \begin{array}{l} < 0 \text{ east of the 500-hPa ridge} \\ > 0 \text{ east of the 500-hPa trough} \end{array} \right.
\]

Thus, there is rising motion above the surface low and subsidence above the surface high. This pattern of vertical motion is just what is required to produce the thickness tendencies.
between 500 to 1000 hPa layer induced by the geostrophic relative vorticity advection. Note that positive vorticity is associated with negative geopotential deviations, as vorticity is proportional to the Laplacian of geopotential.

Thus, the vertical motion maintains a hydrostatic temperature field (i.e., a field in which temperature and thickness are proportional) in the presence of differential vorticity advection. Without this compensating vertical motion, either the vorticity changes at 500 hPa could not remain geostrophic or temperature changes in the 500-1000 hPa layer could not remain hydrostatic.

**Fig. 6.12** Schematic 500-hPa height contours (solid lines), isotherms (dashed lines), and vertical motion field ($w > 0$ dash-dot lines, $w < 0$ dotted lines) for a developing synoptic-scale system. Upward motion occurs where vorticity decreases moving left to right along an isotherm, and downward motion occurs where vorticity decreases moving left to right along an isotherm.
THE Q VECTOR

From the same set of QG prediction equation in a beta-plane, we can derive

$$\frac{D_t}{D_t} \left( f_0 \frac{\partial u_g}{\partial p} \right) = -Q_2 + f_0^2 \frac{\partial u_a}{\partial p} + f_0 \beta y \frac{\partial v_g}{\partial p} \tag{6.47}$$

$$\frac{D_t}{D_t} \left( \frac{R \partial T}{P \partial y} \right) = +Q_2 + \sigma \frac{\partial \omega}{\partial y} + \frac{\kappa}{P \partial y} \frac{\partial J}{\partial y} \tag{6.48}$$

$$\frac{D_t}{D_t} \left( \frac{f_0 \partial v_g}{\partial p} \right) = +Q_1 - f_0^2 \frac{\partial u_a}{\partial p} - f_0 \beta y \frac{\partial u_g}{\partial p} \tag{6.49}$$

$$\frac{D_t}{D_t} \left( -\frac{R \partial T}{P \partial x} \right) = -Q_1 - \sigma \frac{\partial \omega}{\partial x} - \frac{\kappa}{P \partial x} \frac{\partial J}{\partial x} \tag{6.50}$$

where

$$Q_2 \equiv -\frac{R \partial V_g}{p \partial y} \cdot \nabla T \tag{5.45a}$$

$$Q_1 \equiv -\frac{R \partial V_g}{p \partial x} \cdot \nabla T \tag{5.45b}$$

$Q_2$ tends to destroy the thermal wind balance between the vertical wind shear of the zonal wind and the meridional temperature gradient. Similarly, $Q_1$ destroys the thermal wind balance between vertical shear of meridional wind and the zonal temperature gradient. Therefore, an ageostrophic (vertical) circulation is thus required to keep the flow in approximate thermal wind balance.

Some further manipulations on equations (6.47-50) result in the Q vector form of the $\omega$ equation:

$$\left( \sigma \nabla^2 + f_0 \frac{\partial^2}{\partial p^2} \right) \omega = -2 \nabla \cdot Q + f_0 \beta \frac{\partial v_g}{\partial p} - \frac{\kappa}{p} \nabla^2 J \tag{6.53}$$

where

$$Q = (Q_1, Q_2)$$

A convergent (divergent) Q forces ascent (descent).
In a Cartesian coordinate system in which the $x$ axis is parallel to the local isotherm with cold air on the left, the Q vector can be expressed in a simpler form:

$$Q = -\frac{R}{p} \left( \frac{\partial T}{\partial y} \right) \left( \frac{\partial v_g}{\partial x} \mathbf{i} - \frac{\partial u_g}{\partial x} \mathbf{j} \right) = -\frac{R}{p} \left( \frac{\partial T}{\partial y} \right) \left( \mathbf{k} \times \frac{\partial \mathbf{V}_g}{\partial x} \right)$$ \hspace{1cm} (6.55)

Thus, the Q vector can be obtained by evaluating the vectorial change of $\mathbf{V}_g$ along the isotherm (with the cold air on the left), rotating this change vector by $90^\circ$ clockwise, and multiplying the resulting vector by $|\partial T / \partial y|$. 

The Q-vector formulation has the following advantages:

1. The forcing functions on the rhs of equation ?? can be evaluated on an isobaric surface with only the information of $\Phi$ and $T$. On the other hand, in the “traditional” formulation, differential vorticity advection must be computed from height fields at two levels.
2. The forcing function are Galilean invariant;
3. There is no partial cancellation between terms as there often in the traditional formulation;
4. The forcing function is exact under the constraints imposed by the quasigeostrophic approximation; no term has been neglected;
5. Q-vectors may be plotted on analyses of height and temperature to obtain a clear representation of vertical motion and ageostrophic wind.

The QG $\omega$ equation written in terms of the Q vector may be expressed in isentropic coordinates. This is particularly useful when there are strong gradients in potential temperature: Increased resolution is attained on isentropic surfaces, which are well resolved in soundings.
\[ \nabla_\theta \cdot \frac{\kappa \pi}{p} \nabla_\theta \omega^* - f_0^2 \frac{\partial}{\partial \theta} \left( \frac{\partial \theta}{\partial p} \frac{\partial \omega^*}{\partial \theta} \right) = -2 \nabla_\theta \cdot Q - \beta \frac{\partial \pi}{\partial x} \]

where

\[ Q = - \begin{pmatrix} \frac{\partial v_g}{\partial x} \cdot \nabla_\theta \pi \\ \frac{\partial v_g}{\partial y} \cdot \nabla_\theta \pi \end{pmatrix}, \text{ where } \pi \equiv C_p \left( \frac{p}{p_0} \right)^{R/C_p} = \frac{\partial M}{\partial \theta} \]

\[ \omega^* = \left( \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \right) p \]