Finite-amplitude Lagrangian-mean wave activity diagnostics
applied to the baroclinic eddy life-cycle

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Lagrangian-mean wave activity diagnostics are applied to the non-linear baroclinic eddy life-cycle in a simple general circulation model of the atmosphere. The growth of these instabilities through baroclinic conversion of potential temperature gradients and the subsequent barotropic decay can exhibit two distinct life-cycles. One life-cycle results in equatorward propagation of the growing eddy, anti-cyclonic wave-breaking and a poleward shift of the mean jet. The second life-cycle is distinguished by limited equatorward propagation and cyclonic wave-breaking on the poleward flank of the jet. Using a conservative finite-amplitude, Lagrangian-mean wave activity (negative pseudomomentum) to quantify wave growth and propagation, reveals more details about the life-cycles than could be discerned from eddy-kinetic energy (EKE) or other Eulerian metrics. It is shown that the distribution of pseudomomentum relative to the latitude of the axis of the jet can be used to provide a clear distinction between the two life-cycles at an early stage in their development and hence a prediction for the subsequent shift of the jet. This suggests that the distribution of pseudomomentum may provide some predictability for the atmospheric annular modes.
1. Introduction

The breaking of atmospheric waves has long been recognized as an important and complex process, crucial to the advancement in our understanding of the short term weather forecast, seasonal and intra-seasonal predictability and the general circulation of the atmosphere (McIntyre and Palmer 1983). Simply stated, atmospheric wave-breaking is the irreversible deformation of material surfaces. In the upper troposphere, waves radiated by occluding tropospheric depressions, can break as they propagate upwards and equatorwards, redistributing angular momentum and heat. Many papers have been written about this phenomenon, employing a wide variety of diagnostics, modeling efforts and insightful observations. Recent advances in the Lagrangian theory of wave-mean flow interaction Nakamura and Solomon (2010) provide a new perspective on this classic problem. Using diagnostics based on the distribution of material surfaces, wave-breaking can be more readily quantified.

When a baroclinic jet becomes unstable its subsequent evolution may follow two very distinct paths, first noted by Simmons and Hoskins (1980) and subsequently dubbed Life-Cycle 1 (LC1) and Life-Cycle 2 (LC2) in Thorncroft et al. (1993). The dichotomy between the life-cycles has profound consequences for the evolution of the zonal mean wind. One of the most remarkable things about this instability is that identical perturbations, growing on two nearly indistinguishable, zonally symmetric background states, may evolve along these very distinct paths (Thorncroft et al. 1993). Hartmann and Zuercher (1998) demonstrated that a slight change in background shear could induce an abrupt transition from LC1 to LC2. This investigation will examine the results of a set of experiments, which explore a parameter space spanning this transition, using the combined diagnostic tools of a Lagrangian-mean, finite-amplitude wave activity and the Eliassen-Palm (EP) flux. Previous studies have either used EP-flux and linear wave theory (Hartmann and Zuercher 1998; Limpasuvan and Hartmann 2000), or finite-amplitude diagnostics whose flux cannot be directly related to the tendency of the mean flow (Magnusdottir and Haynes 1996; Esler and Haynes 1999). Because this new Lagrangian framework is compatible with the familiar transformed Eulerian-mean, it
provides a simple way to augment a conventional analysis of wave-mean flow interaction.

The life-cycles have been characterized in various different ways. Simmons and Hoskins (1980) recognized that LC2 exhibits much slower decay of eddy kinetic energy than LC1. This difference in persistence was also noted by Thorncroft et al. (1993) when looking at wave activity. There is also a pronounced difference in the direction of wave propagation between the life-cycles. Hartmann and Zuercher (1998) showed that the enhanced equatorward propagation exhibited in LC1 can be attributed to differences in the refractive indices associated with the zonal mean background state. Magnusdottir and Haynes (1996) employed a finite-amplitude wave activity and noted that the meridional excursion exhibited during LC1 was partly non-linear advection and not completely a wave propagation phenomenon. These differences in the propagation of waves and the dissipation of wave activity result in a poleward shift of the jet in LC1 relative to LC2, as noted by many authors. This jet shift is associated with a meridional dipole in angular momentum, which has a barotropic structure and projects onto the atmospheric annular mode (Limpasuvan and Hartmann 2000). This paper will complement these observations with a demonstration that such a dipole is exhibited in the distribution of angular pseudomomentum as well, a result which is not evident from the distribution of EKE.

The annular modes represent the primary pattern of atmospheric variability in the extratropics, whose phase has major influences on regional climate. There is no consensus on the fundamental, underlying cause for the variability of these pressure and wind patterns, however improved predictability would be invaluable for medium-range weather forecasting and projections of climate change. Several authors have drawn a connection between wave-breaking and the variability of the annular modes. Esler and Haynes (1999) constructed a wave-breaking index based on the meridional flux of wave activity and used it to relate variability of the waves to the state of the jet. Riviere and Orlanski (2007) relate life-cycles and wave-breaking to the structure of the storm-track and the North Atlantic Oscillation, a pattern closely related to the annular mode. Another recent study by Strong and Magnus-
dottir (2008) also developed an index of wave-breaking and showed that it provided some skill for predicting the phase of the annular mode. In this paper, a new index based on the distribution of wave activity is introduced. This index is shown to clearly distinguish the two life-cycles across a suite of experiments, without the need to calculate any fluxes of momentum.

The next section reviews the diagnostics used to evaluate the experiments in this study. The third section describes the simplified atmospheric model employed to carry out the simulations. The fourth section will detail one pair of experiments which exemplify this transition. The fifth section explores the parameter space spanned by zonal wavenumber and barotropic shear in order to determine any systematic variations in the life-cycle and develop a method for objectively distinguishing them. A discussion of the predictability of jet shifts using Lagrangian-mean diagnostics and comparison with more conventional Eulerian alternatives follows.

2. Diagnostics

This study will employ the finite-amplitude Lagrangian-mean wave activity (negative pseudomomentum) introduced in Nakamura and Solomon (2010), which employs the potential vorticity equivalent latitude \((\phi_{eq})\) as a meridional coordinate (Butchart and Remsberg 1986). The wave activity at time \(t\) is defined as

\[
A(\phi, z, t) = \frac{1}{2\pi a \cos \phi} \left( \int_{Q(\phi_{eq}) \leq q(\phi')} q dS' - \int_{\phi \leq \phi'} q dS'' \right)
\]

where \(q\) is the potential vorticity, the area element \(dS' = a^2 \cos \phi' d\lambda d\phi'\), \(a = 6,378 km\) the radius of the Earth and \(Q(\phi_{eq})\) is the Lagrangian-mean potential vorticity. The equivalent latitude \((\phi_{eq})\) is defined such that the area covered by the two integrals are equal. That is
to say
\[
\int_{Q(\phi_{eq}) \leq q(\phi')} \frac{dS'}{Q(\phi')} = \int q(\phi') dS',
\]
which specifies the value of \( Q \) as a function of \( \phi_{eq} \). The quasi-geostrophic (QG) potential vorticity (PV) is

\[
q = f + k \cdot (\nabla \times \mathbf{v}) + \frac{f}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\rho_0(\theta - \theta_0)}{\partial \theta_0/\partial z} \right)
\]
where \( f \) is the Coriolis parameter, \( k \) the unit upward pointing normal vector, the geostrophic winds \( \mathbf{v}, \theta(\lambda, \phi, z, t) \) is potential temperature, \( \theta_0(z, t) \) is the layer mean potential temperature and \( \rho_0 \propto \exp(-z/H) \).

Consider the conservation of PV, using the non-divergent, geostrophic wind \( \mathbf{v} \)

\[
\frac{\partial q}{\partial t} = -\mathbf{v} \cdot \nabla q = -\nabla \cdot (\mathbf{v} q).
\]
If \( q \) is conserved, the first term in the definition of \( A \) (Eq. 1) is constant following the motion of a parcel of air, and the tendency of wave activity is due only to the flux of PV across the fixed, bounding equivalent latitude

\[
\frac{\partial}{\partial t} (A \cos \phi) = -\frac{1}{2\pi a} \int_{\phi \leq \phi'} \frac{\partial q}{\partial t} dS = -\cos \phi \bar{v} q
\]
here bars denote zonal means and tildes denote deviations from zonal means. Assuming that the waves are geostrophic and ignoring the divergence associated with the gradient in \( f \), it can be shown that

\[
\cos \phi \bar{v} q = \frac{1}{\rho_0} \nabla \cdot \mathbf{F},
\]
where \( \mathbf{F} \) is the QG EP-flux

\[
\mathbf{F} = (F^\phi, F^z) = \rho \cos \phi [-\bar{u} \bar{v}, \bar{f} \bar{v} \theta/(\partial \bar{\theta}/\partial z)]
\]
and \( \nabla \cdot \mathbf{F} = (a \cos \phi)^{-1} \partial(\cos \phi F^\phi)/\partial \phi + \partial F^z/\partial z \). Thus Eq. 5 is approximately

\[
\frac{\partial}{\partial t} (A \cos \phi) = -\frac{1}{\rho_0} \nabla \cdot \mathbf{F}
\]
as demonstrated in Nakamura and Solomon (2010). Furthermore, the QG transformed Eulerian-mean zonal momentum equation on the sphere as derived in Pfeffer (1987) is

\[
\frac{\partial}{\partial t}(\overline{\pi}\cos \phi) = \frac{1}{\rho_0} \nabla \cdot \mathbf{F} + f \overline{v}_R \cos \phi. \tag{9}
\]

So that the Coriolis torque associated with the residual circulation, \( v_R(\phi, z, t) \), balances the tendencies of wave activity and the zonal mean wind

\[
\frac{\partial}{\partial t}(\overline{\pi} + A) = f \overline{v}_R. \tag{10}
\]

In practice the three terms in Eq. 10 tend to be of comparable magnitude, the nature of the balance being determined by the aspect ratio of the wave forcing (Pfeffer 1987). For tall, narrow forcing, in the absence of irreversible processes, the right hand side tends to be negligible, hence Eq. 10 is referred to as a non-acceleration theorem (Nakamura and Zhu 2010a).

3. Model

The model used in this set of experiments is an atmosphere only model, which employs the GFDL spectral dynamical core on thirty vertical sigma levels specified as

\[
\sigma_k \equiv p(k)/p_s = \exp(\gamma(30 - k)/30), \quad \text{for} \quad k = 0, \ldots, 30 \tag{11}
\]

where \( p(k) \) is pressure on the \( k \)th level, \( p_s \) the surface pressure and \( \gamma = \log(2 \times 10^{-2}) \).

There is no external heating or dissipation during the baroclinic eddy life-cycle, except for hyperdiffusion to damp the smallest scales. The initial condition is expressed in terms of pseudoheight, which is a linear function of log pressure, \( z = -H \log(p/p_0) \), where the scale height \( H = 7.5\, \text{km} \) and reference pressure \( p_0 = 1\, e5\, \text{Pa} \). The initial zonal wind is specified to resemble the experiments in Hartmann and Zuercher (1998)

\[
u(\lambda, \phi, z) = u_0(\phi, z) + u_s(\phi) + u_m(\lambda, \phi, z) \tag{12}\]
where \( u_0 \) is a baroclinic jet, \( u_s \) a barotropic jet and \( u_m \) the velocity anomaly associated with the normal mode of zonal wavenumber \( m \) scaled to a standard amplitude for each case. The specified profile for the baroclinic jet is

\[
u_0(\phi, z) = U_0 \sin^3(\pi \sin^2 \phi)(z/z_T \exp(-z^2/z_T^2 - 1/2)), \quad \phi > 0 \quad (13)
\]

here the parameters are \( U_0 = 45 \, m \, s^{-1} \) and \( z_T = 13 \, km \). The barotropic component of the initial wind field is

\[
u_s(\phi) = U_s \left\{ \exp \left[ -\left( \frac{\phi - \phi_e}{\Delta \phi} \right)^2 \right] - \exp \left[ -\left( \frac{\phi - \phi_p}{\Delta \phi} \right)^2 \right] \right\} \quad (14)
\]

where \( \phi_e = 20^\circ \), \( \phi_p = 50^\circ \) and \( \Delta \phi = 12.5^\circ \). The barotropic shear parameter \( U_s \) is systematically varied amongst the experiments in order to explore its effect on the evolution of the normal mode perturbation. The temperature and surface pressure are derived from geostrophic and hydrostatic balance such that the temperature is in thermal wind balance with the initial zonal wind, following the method described in Polvani and Esler (2007). The lower boundary of the model is specified as a geopotential surface \( \Phi(\phi, p_s) = 0 \), which allows the surface pressure to vary in time. Geopotential height is calculated as

\[
\Phi(\phi, p) = \int_p^{p_0} \frac{RT_r(p')}{p'} dp' - \int_0^{\phi} (af + u \tan \phi') ud\phi' \quad (15)
\]

where the reference temperature at the equator is

\[
T_r(p) = T_s + \frac{\Gamma_0}{(z_T^{-\alpha} + z^{-\alpha})^{1/\alpha}} \quad (16)
\]

for \( T_s = 300K \), \( \Gamma_0 = -6.5K \, km^{-1} \) and \( \alpha = 10 \).

4. One Transition Case

In this section one pair of experiments which clearly depict the transition from LC1 to LC2 will be examined in detail. The initial states for these two experiments are very similar.
Both have a small zonal wavenumber 6 perturbation to the zonally symmetric background flow. The only difference between the two is the value chosen for the barotropic shear parameter $U_s$. The LC1 case, which has slightly lower shear $U_s = 9.5\text{ m s}^{-1}$, is associated with anti-cyclonic wave-breaking. Anti-cyclonic wave breaking refers to the NE-SW tilting of troughs on the equatorward flank of the jet, as illustrated with maps of $\Theta$ on PV surfaces in Thorncroft et al. (1993) (Fig. 7). Whereas, the LC2 case, in which $U_s = 10\text{ m s}^{-1}$, exhibits so-called cyclonic wave breaking. Cyclonic breaking involves the cutting-off of nearly circular vortices on the poleward side of the jet, which become homogenized in their interiors and tend to be long-lived (Thorncroft et al. (1993) Fig. 9). As we see in Fig. 1 the two cases have nearly identical initial zonal mean wind distributions (left column), differing from each other by less than $1\text{ m s}^{-1}$ within the domain (bottom-left). However, this subtle difference in initial conditions results in very different zonal mean wind fields on day 30 (center panel), at the end of the simulation. The evolution of the high shear case (LC2), is best described as a sharpening of the jet with a slight equatorward shift (top-right). The low shear case (LC1) results in a significant poleward shift of the jet, the jet also narrows in latitudinal extent and intensifies (middle-right). The difference between the two final states may be characterized as a dipole, with a node at $55^\circ$, whose structure is fairly uniform below $15\text{ km}$ (bottom-middle). These are basic features of the life-cycles, which have been discussed in previous papers and here will be explained in the context of wave activity.

In the past decade a number of studies have investigated the vacillation of the extra-tropical eddy-driven jet as an equivalent barotropic process. Limpasuvan and Hartmann (2000) showed that the annular modes, the principal mode of variability in the extra-tropical circulation, exhibit largely barotropic structure in both the height field and zonal winds. This variability is manifest as a meridional dipole structure associated with tropospheric wave-breaking, with centers of action located between $30-40^\circ\text{N}$ and $50-60^\circ\text{N}$. Subsequent work by Lorenz and Hartmann (2001) and Vallis et al. (2004) underscored the equivalent barotropic nature of the annular modes and the potential for understanding their variability with
stochastic models driven by eddy forcing.

To illustrate this perspective and draw a connection between these life-cycle experiments and the detailed analysis of observations and models undertaken by previous investigators, we consider the barotropic angular velocity in Fig. 2 (left panels). The barotropic angular velocity is simply a pressure weighted vertical average of the zonal wind

\[
\langle \overline{u}(\phi, t) \rangle \cos(\phi) = \frac{1}{p_s} \int_0^{p_e} \overline{u}(p, \phi, t) \cos(\phi) dp. \tag{17}
\]

Results from the LC2 experiment (top) show the slight equatorward shift of the mean westerlies, with the development of zonal mean easterlies on the poleward flank. LC1 (middle) shows a pronounced poleward shift around day 12 and progressive strengthening and sharpening of the jet between days 15 and 20. The difference between the mean winds in these two life-cycles (bottom) exhibits a dipole structure reminiscent of the difference between extreme states of observed annular modes.

Recent investigations into the annular modes have emphasized the role of wave-breaking. It now appears clear that the high phase of the annular mode is associated with anti-cyclonic breaking of synoptic scale waves (Benedict et al. 2004; Riviere and Orlanski 2007; Strong and Magnusdottir 2008). Employing the Lagrangian-mean wave activity to visualize the role of waves in these experiments provides a new perspective on the evolution of the circulation in this wave-breaking process. The barotropic wave activity \([A](\phi_{eq}, t) \cos(\phi_{eq})\) is plotted in the right panels of Fig. 2. LC2 (top) and LC1 (middle) have similar distributions of angular pseudomomentum prior to day 15, with a localized maximum centered around 50N. LC1 shows enhanced equatorward propagation between days 15 and 20, then steady decay of \([A]\) at all latitudes as the simulation progresses. In contrast, LC2 exhibits a brief period of transience in wave activity around day 15, followed by a poleward shift of the maximum in \([A]\) and steady increase until day 20. This new quasi-equilibrium state of wave activity persists for the duration of the experiment. These differences are comparable to the distinction between LC1 and LC2 made by Simmons and Hoskins (1980), which emphasized the fact
that both cases attain similar maximum levels of eddy-kinetic energy, but in the LC2 case the EKE dissipates much more slowly. However, the meridional propagation of the eddies is not as clear from EKE, which exhibits only a single maximum that tends to follow the latitude of maximum winds (not shown). Using this Lagrangian-mean finite-amplitude wave activity, the meridional dipole in angular pseudomomentum mirroring that of velocity (lower-right), can be discerned in the difference between the two life-cycles. These opposing dipoles illustrate the exchange of pseudomomentum for angular momentum underlying this wave-mean flow interaction in a way that cannot be reproduced from linearized, Eulerian metrics that do not possess an exact non-acceleration theorem (Solomon and Nakamura 2012).

We have seen that the distinction between the two cases is clearly depicted by the temporal evolution of wave activity. In Fig. 3 the time average distribution of wave activity is shown on the left. The two cases have similar distributions of mean wave activity, however it is apparent that the LC2 case has larger time averaged wave activity and the LC1 case has broader meridional extent. In the panels on the right, the temporal evolution of wave activity for the two cases are illustrated by concatenating thirty snapshots of wave activity in the meridional plane. This space-time diagram illustrates that the growth of wave activity between days 6 and 12 is very comparable between the two cases, attaining nearly identical maximum values in the upper-troposphere. However, the subsequent evolution follows extremely different paths. In the LC1 case (bottom-right) the waves begin to decay around day 12 and continue to diminish at all vertical levels as the simulation proceeds. In contrast, the LC2 case (top-right) shows some transience in the mid-troposphere around day 12, but then the tropospheric waves grow again until they have equilibrated around day 24. Additionally, the secondary maximum of wave activity in the stratosphere, which emerges in both simulations around day 12, grows and descends in LC2 while it has all but vanished by day 20 in LC1. This figure indicates that the crucial difference in the evolution of these simulations must manifest itself between days 10 and 15, so in this interval we will examine the dynamics more closely, using the combined diagnostic power of $A$ and the EP-flux.
Fig. 4 shows five snapshots of wave activity between days 6 and 18 of each experiment. In the top row $A$ is plotted in the meridional plane for the LC2 case with the EP-flux vectors overlain. Plotted beneath each snapshot is the near surface potential temperature gradient, which represents the wave source. On day 6 the waves are still negligibly small, but a region of eddy heat flux between 40N and 60N coincides with the latitudes of maximum magnitude temperature gradient. By day 9 a measurable density of pseudomomentum has accumulated between 5km and 10km, which continues to expand on day 12. The temperature gradient has developed a bimodal structure although upward flux persists throughout the extratropics with divergence aloft. On day 15 the maximum in wave activity has shifted toward the pole and the EP-flux is now poleward everywhere except on the equatorward flank of the secondary maximum in wave activity developing between 20km and 25km. Day 18 depicts very strong wave activity north of 40N in both the troposphere and stratosphere. The orientation of this locus for wave activity, directly above a region of enhanced meridional temperature gradient and upward EP-flux, contributes to the persistence of waves in the LC2 case.

Corresponding snapshots for the LC1 case are plotted in the middle panels and the difference between the two cases is displayed in the lower panels. Until day 12 there is little difference visible between the two cases in either wave activity or surface temperature gradients. On day 12, although the distributions of wave activity are still nearly identical, there is a greater heat flux at high latitudes in the LC2 case. By day 15 a perceptible difference in the density of pseudomomentum in the troposphere near 60N has emerged. Furthermore, the EP-flux on day 15 in the LC1 case is everywhere equatorward, opposite to that of the LC2 case. On day 18 the wave activity in the LC1 case has diminished dramatically, strong equatorward fluxes below 15km continue to erode the remaining wave activity at high latitudes. At this point the difference in wave activity density between the two cases has also become visible in the stratosphere and a sizable discrepancy in surface temperature gradients has emerged.
On day 15 the two life-cycles exhibit substantial differences in the distribution and tendency of momentum, which can be estimated by a variety of methods that we compare here. In the top row of Fig. 5 the tendency of wave activity is estimated by taking the difference between its density on days 16 and 14. Values from the LC2 experiment (left-top) show a dipole structure in the vertical with waves decaying below 15 km and growing above, consistent with the vertical propagation of waves at this stage of the life-cycle discussed by Thorncroft et al. (1993). The tendency of wave activity in the LC1 case (center-top) displays largely similar structure to the high shear case, however the magnitudes are very different. To illustrate more clearly the discrepancy in their structure, the difference between their tendencies is plotted on the right. We find an increasing deficit of wave activity in the LC1 case, relative to the LC2 case, throughout the high latitudes of the free atmosphere. The second row of Fig. 5 shows the instantaneous zonal mean poleward flux of PV, which in theory should be the negative of the wave activity tendency. The agreement in the spatial structure of these two quantities reassures us that our understanding of the quantity $A$ is valid. In addition to these basic confirmations, we can also see a meridional dipole structure in the difference between these instantaneous tendencies on day 15.

In the third row of Fig. 5 we see the EP-flux divergence, which agrees well with the PV flux. Overlaid are the EP-flux vectors, which help to interpret their divergence. In the LC2 case, poleward flux at 35N diverges strongly both upward and downward suggesting reflection of waves near 10km. In the LC1 case, the pattern is reversed with equatorward flux at 60N, diverging in the vertical again around 10km. It is worth noting here that these three measures of the tendency of momentum rely on very different assumptions and approximations, yet they exhibit reasonable, quantitative agreement. In particular, the fact that the quasi-geostrophic EP-flux convergence matches the tendency of $A$, underscores the point that this Lagrangian framework is consistent with the familiar, flux based, Eulerian, linearized diagnosis. The lower panels show the Coriolis torque associated with the residual circulation. Note that this term is the same order of magnitude as the other tendency terms.
and typically opposes them. Both cases indicate poleward circulation at upper levels and
an equatorward branch near the surface. The difference between the two shows a stronger
circulation near the surface north of 40\textdegree N in the LC1 case, which contributes to the difference
in surface potential temperature gradients.

Previous investigations have employed linear wave theory and refractive indices to dis-
tinguish the evolution of the two life-cycles (e.g. Hartmann and Zuercher (1998)). Planetary
waves tend to propagate toward regions of high refractive index, $n^2$, and can be reflected
where the index vanishes (Karoly and Hoskins 1982)

$$n^2(\phi, z, t) = \frac{\partial q(\phi, z)/\partial \phi}{a(\mu - c_p)} + N^2 m^2$$  \hspace{1cm} (18)

where the Brunt-Vaisala frequency $N^2 = (g/\theta_0)\partial \theta_0/\partial z$ and $c_p$ is the phase velocity of the
wave. The dominant term in the refractive index is proportional to the meridional gradient
of zonal-mean PV, so in regions where the PV gradient vanishes or becomes negative, wave
propagation is suppressed according to linear theory. Great insights into wave-mean flow
interaction have been developed within the framework of refraction and critical layer theory
(e.g. Killworth and McIntyre (1985)), however linear theory cannot address all aspects of
the baroclinic life-cycle once wave-breaking and mixing begin to significantly modify the
mean state (Hartmann and Zuercher 1998). To investigate the changing PV gradients in
the evolution of the life-cycles we will consider both the zonal-mean and Lagrangian-mean
PV in Fig. 6. By differentiating Eq. 1 twice with respect to latitude (enforcing $\phi_{eq} = \phi$),
it can be shown that the two mean-PV gradients differ due to a contribution from the wave
activity

$$\frac{\partial q}{\partial \mu} = \frac{\partial Q}{\partial \mu} + \frac{\partial^2 A \cos(\phi)}{\partial \mu^2}$$  \hspace{1cm} (19)

where $\mu = \sin(\phi)$, as derived in Nakamura and Zhu (2010a). The zonal-mean PV gradient is
not a conserved quantity, because of the contribution from the curvature of the wave activity
profile it can be modified dramatically in the presence of waves. The Lagrangian-mean PV
is conserved for adiabatic, frictionless flow, so its gradient tends to evolve more slowly and
systematically, responding only to irreversible mixing and other non-conservative processes.
Fig. 6 shows snapshots of the meridional gradients of the two mean PV fields for the initial and final day of each experiment. On day 1 (left) the fields all appear indistinguishable, since the initial conditions for each experiment are very similar and essentially zonally symmetric. We see a localized maximum of the PV gradient near 45N at approximately 8km, close to the axis of jet. This distribution will tend to focus waves into the jet, making the jet a strong waveguide for the growing eddies. The final distribution of PV in each case shows reduction in the meridional extent of the region of strong gradients, associated with sharpening of the jets. The Lagrangian-mean PV increases monotonically with equivalent latitude by construction and indicates sharper gradients than the zonal-mean. The Eulerian, zonal-mean PV gradients are reversed on the flanks of the jet by the end of the simulation and have somewhat smaller maximum value.

The set of panels farthest right in Fig. 6 show the difference between the initial and final distributions of PV gradients (day 30 - day 1). Here we see that the two life-cycles result in oppositely signed dipole anomalies in the upper troposphere, with a node near the initial jet axis. These patterns reflect the fact that the region of enhanced PV gradient is closely associated with the jet, which shifts oppositely in latitude over the course of the two life-cycles. The two averages portray fairly consistent changes, although the dipole patterns are narrower in equivalent latitude, due to the sharper final $Q_{\phi_{eq}}$ distributions. The largest discrepancies are seen at high latitudes in the LC2 experiment, where waves continue to contribute to the zonal-mean PV gradients on day 30. This dipole pattern in PV gradient depicts a change in the curvature of the mean PV near the initial axis of the jet. An alternate metric for the shift of the jet can then be calculated both from the Lagrangian and zonal means.

$$\Delta PV_{\phi\phi}(t) = \frac{1}{a^2} \int_5^{10} \frac{\partial^2}{\partial \phi^2} [PV(z, t) - PV(z, 0)] \big|_{45N} dz$$

The panel at the bottom of Fig. 6 shows $\Delta PV_{\phi\phi}$, the change in the mean tropospheric curvature of $PV$ at the initial jet axis as a function of time, for both experiments. The Lagrangian-mean PV (solid curves) indicates enhanced curvature in both cases until day 13,
when the two life-cycles diverge from one another. After day 13 the PV gradient on the pole-
ward flank of the jet begin to decrease in LC2, ultimately resulting in the observed dipole and
decreased curvature relative to the initial state. In contrast, the LC1 change in PV curvature
remains positive, rising to its peak value after day 22. The corresponding diagnosis using \( \bar{q} \)
is plotted in dashed curves, indicating similar conclusions about the temporal development
but smaller signal in the LC1 case. In the next section we will employ \( \Delta PV_{\phi\phi} \) to characterize
the jet shift resulting from the two life-cycles and utilize the association between maximum
\( Q(\phi_{eq}) \) gradients and the jet axis to define a wave index.

5. The effects of increasing barotropic shear and varying zonal wavenumber

In Fig. 7 the time evolution of the global mean wave activity is plotted for three selected
wavenumbers at a range of barotropic shears. Within this range of values for \( U_s (0 - 12 \text{ m s}^{-1}) \)
we see a clear progression in the behavior at these wavenumbers (\( m = 5, 6, 7 \)). These plots
are similar to those of Hartmann and Zuercher (1998), who plotted eddy-kinetic energy
for a wavenumber 6 disturbance growing in a similar background flow and calculated the
relevant energy conversion terms in order to diagnose the evolution of the flow. The initial
stage of exponential growth is due to baroclinic conversion through eddy-heat flux and is
very similar for both life-cycles. The subsequent barotropic decay phase characterized by
large eddy-momentum fluxes is very different in the two life-cycles. LC2 exhibits much less
barotropic conversion during this second phase of the life-cycle (Hartmann and Zuercher 1998).

On the left, timeseries for wavenumber five, clearly depict the conventional LC1 progres-
sion. The initial period of exponential growth is clear, followed by a wave breaking instability,
which results in a dissipation of wave activity until a new quasi-steady state is achieved. This
dissipation of wave activity is associated with an exchange of pseudomomentum for angular
momentum, resulting in the observed changes in the zonal mean wind (Fig. 1). The growth of the instability is delayed with increasing shear (redder colored curves indicate higher $U_s$). Also the maximum global mean wave activity attained generally increases with greater shear. The tendency of decreasing growth rate with rising shear, observed at $m = 5$ persists for all higher wavenumbers examined in this study. The maximum global mean wave activity attained is a complicated story, which will not be addressed in this study.

The cases examined in detail were wavenumber six perturbations, which are plotted in the middle. Here we see the transition from LC1 to LC2 as the $U_s = 9.5 m s^{-1}$ (yellow) diverges from the $U_s = 10 m s^{-1}$ (orange) after day 13. The LC2 cases, initialized with $U_s \geq 10 m s^{-1}$, exhibit less dissipation of wave activity after the peak of the life-cycle than the experiments with slightly lower barotropic shear. Even the very low shear cases (bluer colors), which attain only half the peak amplitude of the LC2 cases, exhibit greater dissipation in the immediate aftermath of their exponential growth phase. Finally, the $m = 7$ case is plotted in the right panel. The two experiments with lowest initial barotropic shear still exhibit LC1. However, all $m = 7$ cases with $U_s \geq 4 m s^{-1}$ develop as LC2. These $m = 7$ cases attain lower maximum amplitudes than the corresponding longer wavelength perturbations and peak earlier, with their exponential growth phase ending prior to day 10 for almost all cases. The LC2 experiments can actually exhibit increasing pseudomomentum after the period of normal mode growth, as seen in these $m = 7$ timeseries, although that growth is not systematic nor a large fraction of the total wave activity. All simulations for higher wavenumbers exhibited the LC2, with smaller peak values of $A$ and little or no dissipation.

In order to objectively distinguish the two life-cycles, for every one of these simulated cases, we cannot simply rely on the magnitude of wave activity alone. The maximum amplitude of the waves decreases dramatically with increasing wavenumber, so the temporal and spatial-domain averaged wave activity convolves the issues of persistence and peak amplitude. The fundamental distinction between the LC1 and LC2 cases is the extent of wave-breaking on the equatorward flank of the jet, as illustrated by Thornicroft et al. (1993).
In LC1 filaments of PV are sheared out to the south of the jet where they dissipate over time. The LC2 case is distinguished by a general absence of this equatorward wave propagation and subsequent breaking. So we can define an index which quantifies the fraction of wave activity found on the equatorward side of the jet axis

\[
WI(t) = \frac{\int_0^z J(z,t) \int_0^A A(\phi_{eq}, z, t) d\phi_{eq} dz}{\iint_A d\phi_{eq} dz}
\]  

(21)

where the axis of the jet is defined as \( J(z, t) = \max_{\phi_{eq}} \{\partial Q/\partial \phi_{eq}\} \), the equivalent latitude of maximum PV gradient. This index is similar to a quantity considered by Magnusdottir and Haynes (1996), however they employed a fixed latitude to distinguish the two flanks of the jet.

In Fig. 8 both \( WI \) and the domain averaged wave activity are plotted for each case examined. In the left column we see the time average of each diagnostic, plotted with a distinct color for each case. The mean wave activity (top-left) does not vary consistently for all cases, though a general tendency toward lower values with increasing wavenumber is apparent. A related parameter space, spanned by eddy kinetic energy and frequency, was explored in Riviere and Orlanski (2007), who found that long waves break only anti-cyclonically, while shorter wavelengths may follow either life-cycle. \( WI \) (bottom-left) clearly distinguishes the two life-cycles, with LC1 in red hues and LC2 in bluer shades. An oblique separatrix between the two life-cycles can now be discerned. This parameter-space is partitioned into an LC1 region spanned by low zonal wavenumbers and low values of barotropic shear, and an LC2 region representing the high shear or high wavenumber cases. Transitions from LC1 to LC2 with increasing shear are seen for both \( m = 6 \) and \( m = 7 \), with higher wavenumbers exhibiting only LC2. It is possible that \( m = 5 \) would exhibit a transition to LC2 if even higher values of initial barotropic shear were applied, but no such behavior was observed for the shear range examined.

The panels on the right show timeseries of these two metrics for each experiment. Here the LC2 cases have been highlighted in colors by wavenumber and the LC1 cases are plotted in
black. In the top panel the global mean wave activity illustrates that the LC2 cases cluster tightly according to wavenumber and clearly lack the strong dissipation seen in the LC1 cases. However, the range of mean wave activities for the LC2 cases cannot be distinguished from that of the LC1 cases. $WI$ (bottom) more clearly distinguishes the two cases, with essentially distinct ranges of values after day 10. The wavenumber 5 cases do not break until somewhat later, rising to values of $WI \geq 0.3$ between days 10 and 15, but aside from those few cases the black curves are clearly separated from the the colored curves by day 10. For all the LC2 cases $WI$ diminishes, nearly monotonically, throughout the course of the experiment, asymptotically approaching values below 0.2. Hence the tendency of $WI$ during the eddy growth phase (days 6-10 or so depending on wavenumber and background shear) has opposite signs for the two life-cycles. This fact can be used to predict the subsequent evolution of the instability and the consequent changes in mean wind. Similar indices of wave-propagation and breaking have been constructed in Esler and Haynes (1999); Strong and Magnusdottir (2008), which proved to be useful predictors for the phase of the observed annular mode.

6. Discussion

The ultimate goal of this investigation was to provide a method for predicting the type of life-cycle resulting from a given initial condition. This goal was not attained and questions still remain about the critical factor leading to symmetry breaking in the evolution of this instability. However, as observed by many previous authors (e.g. Thorncroft et al. (1993); Hartmann and Zuercher (1998)), the life-cycles clearly distinguish themselves during the growth and propagation phase, prior to attaining their maximum amplitude and breaking. The fact that this occurs before any significant redistribution of angular momentum or shift of the jet implies that there is some predictive potential to be derived from the distribution of wave activity. The figure below illustrates the shift of the jet after the onset of wave-
breaking, measured by $\Delta PV_{\phi \phi}$, as a function of the tendency of $WI$ a week or more earlier in the life-cycle. Here the tendency of $WI$ is simply calculated as a finite difference, two days before the maximum in wave activity is reached. Because of the difference in growth rates between the long waves ($m = 5, 6$) and the shorter waves ($m = 7, 8, 9$), these characteristic metrics for the life-cycles had to be calculated during different intervals. For the short waves, which grow more rapidly, the tendency of $WI$ was calculated on or prior to day 6 and the PV curvature on day 14. For the long waves $WI_t$ was calculated before day 12 and $\Delta PV_{\phi \phi}$ on day 18. Although the timing of the life-cycles differ with wavenumber, they all suggest about a week of lag between the propagation of waves relative to the jet axis and the ultimate shift of the jet.

In Fig. 9, several clusters of points can be distinguished, the filled circles calculated using the Lagrangian diagnostics and open circles based on a Eulerian metric for comparison. The filled green circles, reflecting LC1 experiments, all plot in the first quadrant. This implies that the equatorward propagation of wave activity early in the life-cycle invariably precedes a poleward shift of the jet by about a week. The filled blue circles, LC2 experiments, generally plot in the third quadrant. This suggests that the negative tendency of $WI$ is a strong indicator of a coming equatorward shift of the jet. The two marginal cases (points in the 2nd and 4th quadrants) are $m = 7$, which fall somewhere between long and short baroclinic waves, so the timing of their life-cycles did not conform to these conventions, although their overall behavior still supports the broader conclusions. Plotted in black are the pair of $m = 6$, transition cases examined in detail. The two cases depict small magnitude and differing signs of $WI$ tendency on day 11 and opposite shifts of the jet by day 18. The fact that the wave propagation is so weak in this pair of transition cases when compared with the tendency of $WI$ for cases that are not as close to the abrupt transition between life-cycles, suggests that a threshold in wave-propagation distinguishes the two life-cycles.

The predictive diagnostic $WI_t$ required several pieces of Lagrangian data. The distribution of wave activity in the equivalent latitude meridional plane is a Lagrangian-mean
diagnostic, which is more robust than its Eulerian counterparts (eddy-kinetic energy, linear zonal-mean pseudomomentum) (Solomon and Nakamura 2012). The jet latitude $J(z,t)$ has been defined in terms of the maximum gradient of Lagrangian-mean PV, primarily for the practical purpose of determining the location of the jet in terms of equivalent latitude. Also, the eddy propagation phase was determined based on the timing of maximum of wave activity for each life-cycle. To determine how important this Lagrangian information is we considered an Eulerian diagnostic (open circles), plotted for comparison with $W_I_t$. This metric employs eddy-kinetic energy (EKE) in place of wave activity, jet latitude defined as the maximum of the zonal mean wind and timing based on the maximum of mean EKE during the life-cycle. Linear zonal-mean pseudomomentum is unsuitable for this type of analysis because of the vanishing $\bar{q}$ gradient, which makes it impossible to calculate domain averages since the linear pseudomomentum becomes infinite. From Fig. 9 we can see that the Eulerian diagnostic is not able to distinguish the two life-cycles, with the majority of LC2 cases exhibiting increasing EKE south of the latitude of maximum winds (open circles in the second quadrant). The timing of maximum EKE is fairly consistent with wave activity, although $A$ tends to develop more smoothly and shows less dissipation in the LC2 cases (not shown). The distribution of eddies relative to the maximum PV gradient seems to be a more reliable discriminant between the two life-cycles than the distribution with respect to the maximum winds. This supports the perspective that it is the mixing of PV by eddies and resultant modification of the PV gradient which is more fundamental to the dichotomy between life-cycles than the flux of momentum.

7. Conclusions

The use of Lagrangian-mean wave activity provides insights into the evolution of wave activity in these life-cycle experiments. The use of a conservative diagnostic allows easier quantification of the spatio-temporal evolution of wave activity density than conventional
Eulerian diagnostics. Measures like eddy kinetic energy only provide robust information in the domain average. Lagrangian-mean finite-amplitude wave activity can illustrate the specific regions of the domain whose circulation is being modified by the presence of waves as well as reproduce prior conclusions about the temporal evolution of domain average pseudomomentum. Domain average $A$ clearly depicts the initial period of exponential growth common to both life-cycles and the subsequent dichotomy in trajectories, with dissipation in LC1 and persistence in LC2. The spatial distribution explains this difference, underscoring the equatorward breaking of waves in LC1 and the consequent deceleration on the south flank of the jet, resulting in the apparent poleward shift of the jet. In contrast LC2 evolution involves little or no equatorward wave-breaking, hence no significant redistribution of angular momentum toward higher latitudes.

The index of refraction has been used in previous studies to illustrate the role of critical lines and wave reflection in the evolution of life-cycle experiments Thorncroft et al. (1993); Hartmann and Zuercher (1998); Limpasuvan and Hartmann (2000). Such indices rely heavily on the gradient of zonal-mean potential vorticity, which provides the restoring force that permits Rossby-wave propagation. In the presence of finite-amplitude waves the gradients of PV can become obscured by zonal averaging, making it difficult to interpret the index of refraction and impossible to compute the linear pseudomomentum. In this study we considered the meridional gradient of Lagrangian-mean potential vorticity, which can only be modified through irreversible processes such as mixing. The Lagrangian-mean PV depicts the progressive evolution of a localized maximum in the PV gradient, which is associated with the curvature of the evolving jet. LC1 results in significant mixing on the equatorward flank of the jet, increasing the meridional PV gradient and sharpening the jet, whose mean latitude shifts toward the pole. Whereas, LC2 produces more mixing on the poleward flank of the jet, this also enhances the maximum PV gradient, concentrating it to the south of the original jet axis and diminishing the mean gradient to the north. This provides a context for thinking about baroclinic eddy life-cycles in terms mixing and PV staircases (Dritschel...
Perhaps the most useful outcome of this study is a simple wave index, which utilizes both Lagrangian-mean wave activity and a Lagrangian perspective of jet location, thereby providing a clear distinction between the two life-cycles. We have demonstrated that, in a broad suite of simulations, the tendency of this index can be used to predict the subsequent evolution of the instability and hence the shift of the jet. The use of a Lagrangian perspective of jet axis location, via the maximum PV gradient in equivalent latitude, turned out to be a crucial piece of this construction and makes this diagnostic more applicable to the range of dynamics which might be encountered in observations than a fixed Eulerian prescription of jet latitude (e.g. Magnusdottir and Haynes (1996)). A future study will apply this wave index to observational reanalysis in order to advance our understanding of the role which wave-breaking plays in annular mode variability.

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